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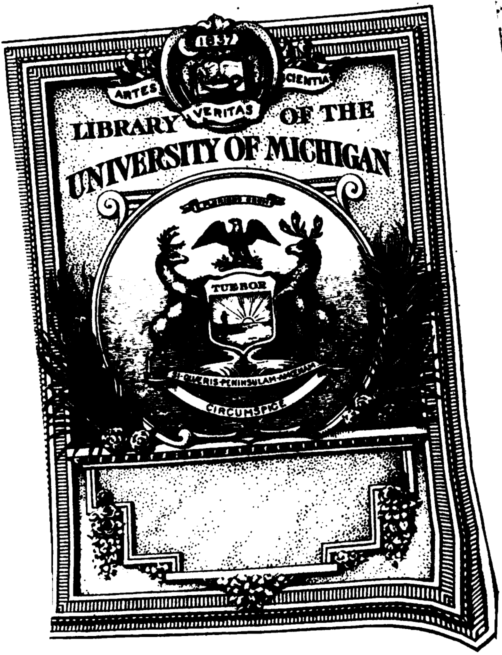

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AN
INTRODUCTION
TO
GEOMETRY;

CONTAINING
The moſt uſeful Propoſitions in EUCLID,
and other Authors.

Demonſtrated in a clear and eaſy Method, for the
Uſe of Learners.

By WILLIAM PAYNE.

THE SECOND EDITION.

LONDON:
Printed by H. HART, Popping's Court, Fleetſtreet.
Sold by T. PAYNE, at the Mews Gate; and
M. HINGESTON, at Temple Bar.
MDCCLXVIII.

THE Critical Review, No. 135, gives the following account of the firſt Edition, viz.

"After having carefully peruſed and conſidered this Treatiſe, we will venture to pronounce it elegant, ſhort, eaſy, and learned. It contains every uſeful propoſition of Euclid's Elements of Geometry (and many others diſcovered ſince his Time) demonſtrated in the moſt conciſe and plain manner poſſible, without any of thoſe ſuperfluous obſtruſe propoſitions, which are ſometimes to be found in Euclid. In ſhort, we think it will prove extremely uſeful to tyros, and all who deſire to learn GEOMETRY without a Maſter."

TO HIS
ROYAL HIGHNESS
THE

DUKE of *YORK*.

SIR,

THEY who are permitted to prefix the names of Princes to treatiſes of ſcience, generally enjoy the protection of a patron, without fearing the cenſure of a judge.

The honour of approaching your Royal Highneſs, has given me many opportunities of knowing that the work which I now preſume to offer, will not partake of the uſual ſecurity. For as the knowledge which your Royal Highneſs has already acquired of GEOMETRY extends beyond the limits of an introduction, I expect not to inform you; I ſhall be happy if I merit your approbation.

An addreſs to ſuch a patron admits no recommendation of the ſcience. It is ſuperfluous to tell your Royal Highneſs that GEOMETRY is the primary and fundamental art of life; that its effects are extended through the principal operations of human ſkill; that it conducts the ſoldier in the field, and the ſeaman in the ocean; that it gives ſtrength to the fortreſs, and elegance to the palace. To your Royal Highneſs all this is already known; GEOMETRY is ſecure of your regard, and your opinion of its uſefulneſs and value has ſufficiently appeared, by the condeſcenſion in which you have been pleaſed to honour one who has ſo little pretenſion to the notice of Princes, as,

SIR,
Your Royal Highneſſes
Moſt obliged,
Moſt obedient,
And moſt humble Servant,
WILLIAM PAYNE.

391649

PREFACE.

THE books containing elements of GEOMETRY *are ſo numerous, and the names by which they are recommended ſo high in mathematical reputation, that a new treatiſe, by an obſcure author, cannot but ſeem to require an apology.*

No man ought to increaſe the crowd of books, without ſome proſpect of increaſing or facilitating knowledge. I do not pretend to boaſt of new diſcoveries, but I hope and expect, to make GEOMETRY *more readily attainable than heretofore, by the method of demonſtration, which, I believe, to be eaſier than in other books of the ſame kind; tho' I am ſenſible, that every man naturally thinks his own mode of explanation the beſt.*

I have ſelected every neceſſary and uſeful propoſition, for the obtaining a knowledge in thoſe ſciences which are dependant upon GEOMETRY, *and have allotted a page to each for the greater eaſe and ſatisfaction of my reader; but how well I have ſucceeded muſt be left to experience.*

INTRODUCTION

TO

GEOMETRY.

1. *GEOMETRY contains the nature and properties of lines, angles, ſurfaces, and ſolids.*

2. *A point is that which has no parts, or magnitude.*

3. *A line is length without breadth.*

4. *A right or ſtrait line, is that which lies evenly between its extremes, or ends.*

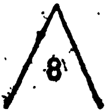
5. *A ſurface is that which has length and breadth, but no thickneſs.*

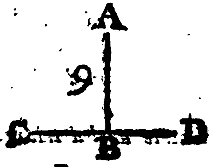
6. *A plain ſurface is that which lies evenly, and agrees with a ſtrait line joining any two points in it.*

7. *A ſolid is that which has length, breadth, and thickneſs.*

8. *A plain angle is an opening or corner made by two ſtrait lines that meet one another.*

9. *When a ſtrait line* AB *ſtanding upon another* CD, *makes angles* ABC, ABD, *on each ſide equal to one another, each of theſe equal angles is called a right angle, and the line* AB *is ſaid to be perpendicular to the line* CD.



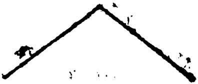


Note. *An angle is commonly expreſſed by three letters; that placed at the angular point being always wrote in the middle.*

10. *An obtuſe angle is that which is greater than a right angle.*

11. *An acute angle is that which is leſs than a right angle.*

12. *Parallels are ſuch ſtrait lines in the ſame plain, which if infinitely continued, or produced, will never meet.*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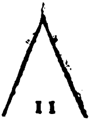
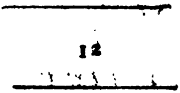




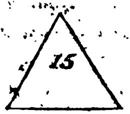
13. *A ſpace bounded on all ſides is called a figure.*

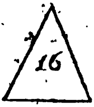
14. *A plain triangle is a figure made by three ſtrait lines.*

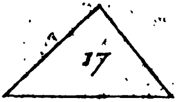
15. *An equilateral triangle is that whoſe three ſides are equal to one another.*

16. *An iſoſceles triangle is that which has only two of its ſides equal to one another.*

17. *A ſcalene triangle is that which has all its ſides unequal.*



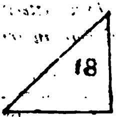


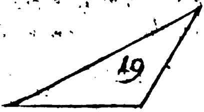


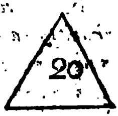
18. *A right angled triangle is that which has a right angle.*

19. *An obtuſe angled triangle is that which has an obtuſe angle.*

20. *An acute angled triangle is that which has every angle acute.*







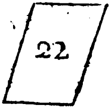
21. *A quadrangle is a figure made by four ſtrait lines.*

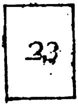
22. *A parallelogram is a quadrangle whoſe oppoſite ſides are parallel.*

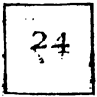
23. *A rectangle is a parallelogram whoſe angles are all right angles.*

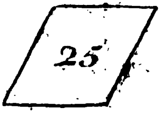
Note. *A rectangle is ſaid to be contained by (or within) any two of its adjacent ſides.*

24. *A ſquare is a parallelogram whoſe ſides are all equal to one another, and its angles are all right angles.*









25. *A rhombus is a parallelogram whoſe ſides are all equal to one another, but its angles are not right angles.*

All figures having more ſides than four, are in general called polygons.

26. *A regular figure is that whoſe ſides are all equal to one another, and whoſe angles are all equal to one another.*

27. *Figures are equiangular, when the angles of the one are equal to the angles of the other, each to each.*

28. *The baſe of a figure, is the line or plain, upon which it is ſuppoſed to ſtand, or be erected.*

29. *The altitude of a figure, is a perpendicular drawn from the top or vertex to the baſe.*

30. *A circle is a figure bounded by a curve line called*

the circumference; every part whereof is equally diſtant from a point within, called the center.

Only ſtrait lines and circles are received into plain Geometry.

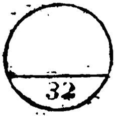
31. *The diameter of a circle is a line paſſing through the center, and terminated by the circumference. The ſemi-diameter is called the radius.*

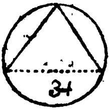
32. *A ſegment of a circle is any part cut off by a ſtrait line.*

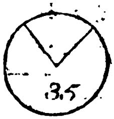
33. *Each ſegment, made by a diameter, is called a ſemicircle.*

34. *An angle is ſaid to be in a ſegment, when lines are drawn from the baſe of the ſegment to any point in the circumference.*

35. *A ſector of a circle, is a part contained between two ſemidiameters and the intercepted part of the circumference.*







36. *Similar ſegments of circles are thoſe which contain equal angles.*

37. *Circles are ſaid to touch, when their circumferences meet, but do not cut one another.*

38. *A tangent to a circle, is a ſtrait line that touches the circumference, and falls without the circle.*

39. *A line is ſaid to be applied or inſcribed in a circle, when its ends are placed in the circumference.*

40. *A figure is ſaid to be inſcribed in a circle, when all its angles are placed in the circumference.*

41. *A circle is ſaid to circumſcribe a figure, when the circumference paſſes thro' all its angular points.*

42. *A figure is ſaid to be deſcribed about a circle, when every ſide touches the circumference.*

43. *A circle is ſaid to be inſcribed in a figure, when the circumference touches every ſide of the ſaid figure.*

44. *One figure is ſaid to be inſcribed in another, when all the angles of the one are placed in the ſides of the other.*

45. *A theorem, is a truth propoſed to be demonſtrated.*

46. *A problem, is ſomething propoſed to be done.*

47. *A corollary, is a truth or theorem, which ariſes from a preceeding demonſtration.*

Axioms are ſelf-evident truths; as,

48. *Things that are equal to the ſame thing, or to equal things, are equal to one another.*

49. *If equal things be added to equal things, the ſums or wholes will be equal.*

50. *If equal things be taken from equal things, the remainders will be equal.*

51. *The halves of equal things are equal, and the doubles of equal things are equal.*

52. *The whole is greater than a part, and equal to all its parts.*

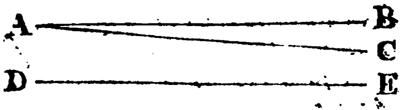
53. *Things that mutually agree, or fill equal ſpaces, are equal to one another.*

54. *All right angles are equal to one another.*

55. *Equal circles have equal ſemidiameters.*

56. *Equal things may be taken for one another.*

57. *Lines that meet one another, are not parallel to one and the ſame line. Thus the lines* A B, A C, *are not both of them parallel to the line* D E.



Poſtu-

Poſtulates, or demands.

58. *A line may be drawn from any one point, to any other point.*

59. *A line may be produced (that is, continued or lengthened) at pleaſure.*

60. *A circle may be deſcribed about any center, and at any diſtance, or with any radius.*

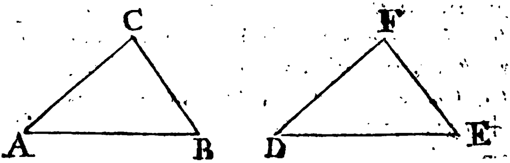
61. *Of the ſigns or characters uſed hereafter.*

+ *ſignifies more.*
— - - - - - *leſs.*
= - - - - - - *is equal to, or are equal to.*
× - - - - - *into. The rectangle contained by two lines, or multiplication by a number. Thus* A B × C D *expreſſes a rectangle contained by the lines* A B, C D; *and* A B × *n, or n* A B *is n times the quantity* A B, *n repreſenting any number whatſoever.*
AB^2 *ſignifies the ſquare of the line* A B.
p. - - - - - *problem.*
th. - - - - - *therefore.*
ag. - - - - - *again.*
con. - - - - - *conſtruction.*
theo. - - - - - *the theorem under conſideration.*

The numbers placed againſt the ſeveral ſteps, or arguments in a demonſtration, are references to preceeding articles.

And the ſtars placed at two ranks of proportionals in article 178, *and ſome others, mark out the equal ratios uſed below.*

62. *Triangles that have two ſides equal to two ſides each to each, and the contained angles equal, are equal to one another, and their remaining parts are equal each to each, namely, the parts oppoſite to equal things.*



Let the triangles A B C, D E F have the ſides B A, A C equal to E D, D F each to each, and the angle B A C equal to the angle E D F: then the triangles are equal, the ſide B C is equal to the ſide E F, and the angles A B C, A C B are equal to D E F, D F E each to each.

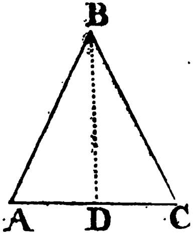
For conceive the triangle A B C applied to D E F, with the point A on the point D; and the line A B on the line D E; the point B will then agree with E, becauſe A B is equal to D E, the line A C will agree with the line D F, becauſe the angle B A C is equal to E D F, and the point C muſt agree with F, ſince A C is equal to D F. Again, as the points B, C agree with the points E, F, the line B C muſt agree with the line E F. - - - - - - - - - 4

Th. the triangle A B C is equal to D E F - - - 53

- - - The ſide B C is equal to the ſide E F - - 53

And the angles A B C, A C B are equal to D E F, D F E each to each. - - - - - - 53

63. *In triangles, oppoſite to equal ſides, are equal angles.*



Let the triangle A B C, have the ſide B C equal to the ſide B A: then the oppoſite angles B A C, B C A are likewiſe equal.

For conceive the line B D drawn to biſect the angle A B C, or to divide it equally.

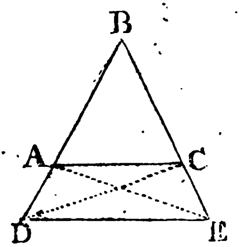
Now the ſide A B is equal to the ſide B C - theo.
The line B D is common to both - -
Th. the ſides A B, B D are equal to C B, B D each to each. - - - - - - - above
And the contained angles A B D, D B C are equal. above
Th. the angle B A C is equal to B C A - - 62

64. Corollary.

An equilateral triangle is equiangular.

63. Otherwiſe.

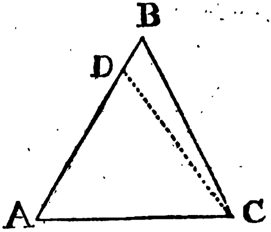
In triangles, oppoſite to equal ſides AB, BC *are equal angles* BAC, BCA.



For produce the equal ſides BA, BC to D and E, make BE equal to BD, or CE equal to AD by problem 2, and join AE, CD.

Now the ſides AB, BE are equal to CB, BD each to each. - - - - - - - - - con.
And the contained angle ABE is common - -
Th. the ſide EA is equal to the ſide DC - - - 62
- - - the angle CEA is equal to ADC - - - 62
And the angle BAE = BCD - - - - 62
Ag. the ſides CE, EA are equal to AD, DC, each to each. - - - - - above
And the contained angles CEA, ADC are equal. above
Th. the angle CAE = ACD - - - - 62
Th. the angle BAC is equal to BCA - - 50

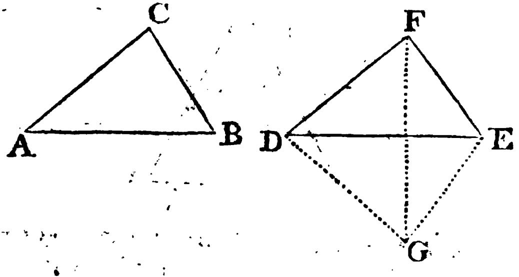
65. *In triangles, oppoſite to equal angles, are equal ſides.*



Let ABC be a triangle, having the angle BCA equal to the angle BAC: then their oppoſite ſides AB, BC are alſo equal.

For if not, let AB be the greater, and in the ſame take AD equal to BC by problem 2, and join DC.
Now the ſide DA is equal to the ſide BC - - con.
Th. the ſides DA, AC are equal to BC, CA each to each
And the contained angles DAC, BCA are equal. theo.
Th. the triangles DAC, BCA are equal - - 62
Or the triangles ADC, ABC are equal - - -
Which is impoſſible - - - - 52
Th. the ſide AB is not greater than the ſide BC. -
By the ſame reaſon, AB cannot be leſs than BC. - -
Th. the ſides AB, BC muſt be equal. - - -

66. *Triangles whoſe ſides are equal to one another each to each, are equiangular, having thoſe angles equal which are oppoſite to equal ſides.*

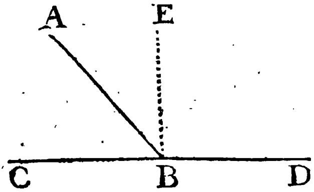


Let the triangles ABC, DEF have the ſides AB, BC, CA equal to DE, EF, FD each to each: then oppoſite to the equal ſides are equal angles.

For with the ſide DE, which is not leſs than DF or FE, make the angle EDG equal to the angle BAC by problem 9, make DG equal to AC or DF by problem 2, and join EG, GF.

Now the ſides ED, DG are equal to BA, AC each to each. - - - - - con.
And the contained angles EDG, BAC are equal. con.
Th. the angle DGE = ACB, GED = CBA, and the ſide EG is equal to the ſide BC - - 62
Th. EG and EF are equal - - - 48
Ag. the angle DGF is equal to DFG - - 63
And the angle FGE is equal to GFE - - 63
Th. the angle DGE = DFE - - - 49
And DG, GE are equal to DF, FE, each to each. above
Th. the angle EDG = EDF, and GED = FED - 62
Th. the angle ACB = DFE, BAC = EDF, and CBA = FED. - - - - - 48

67. *One line meeting another otherwiſe than at the end, makes with it two angles, which taken together, are equal to two right angles.*



Let the line AB meet the line CD in B: then the angles CBA, ABD both together, are equal to two right angles.

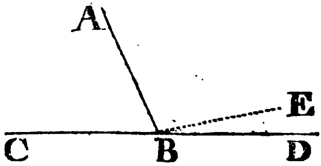
For if the angles CBA, ABD are equal, each is a right angle by the 9th; and if not, draw BE at right angles to CD by problem 6.

Now the angle CBA + ABE - - - - = CBE - 52
Th. - - - - - CBA + ABE + EBD = CBE + EBD 49
Again, the angle - - - ABE + EBD = ABD - 52
Th. - - - - - CBA + ABD - - - - - = CBE + EBD 56
But CBE, EBD are right angles - - con.
Th. the angle CBA + ABD - - - - = two right angles - - - - - - 56

68. Corollary.

All the angles which can be made by lines proceeding from a given point taken together, are equal to four right angles.

69. If from any point in a ſtrait line, two lines be drawn on different ſides making the adjacent angles equal to two right angles, theſe lines will have the ſame direction, and fall into one ſtrait line.

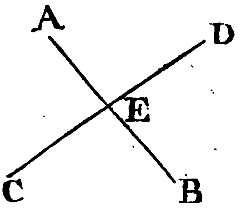


From the point B in the ſtrait line AB, let the lines BC, BD drawn on different ſides of it, make the angle CBA + ABD equal to two right angles: then CBD is a ſtrait line.

For if not, produce CB to E.

Now the angle CBA + ABE = two right angles - 67
And - - - - - - CBA + ABD = two right angles. theo.
Th. - - - - - - CBA + ABE = CBA + ABD - 48
Th. - - - - - - - - - - - ABE = - - - - ABD - - 50
Which is impoſſible - - - 52
Th. CBE is not a ſtrait line - - - - -
Th. CBD muſt be a ſtrait line. - -

70. *The oppoſite angles made by two lines cutting each other, are equal to one another.*

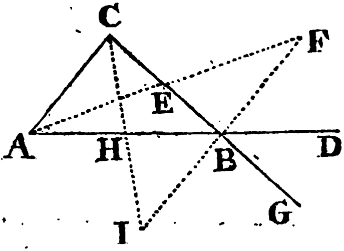


Let the lines AB, CD cut each other in E; then the angle AEC is equal to DEB, and the angle AED is equal to BEC.

For the angle AEC + AED = two right angles - 67
And the angle AED + DEB = two right angles - 67
Th. the angle AEC + AED = AED + DEB - - 48
Th. the angle AEC - - - - = - - - - DEB - 50
Ag. the angle AED + DEB = two right angles - 67
And the angle DEB + BEC = two right angles - 67
Th. the angle AED + DEB = DEB + BEC - - 48
Th. the angle AED - - - - = - - - - BEC - - 50

71. If

61. *If any ſide of a triangle is produced, the outward angle will be greater than either of the inward and remote angles.*

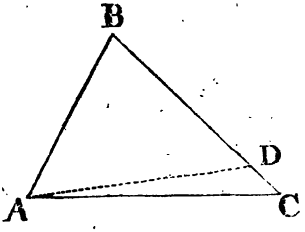


Let ABC be a triangle, whoſe ſide AB is continued to D: then the angle CBD is greater than either of the angles ACB, BAC.

For biſect BC in E by problem 5, draw AE and produce it to F, make EF equal to EA, join FB, and continue CB to G.

Now the ſides CE, EA are equal to BE, EF each to each. - - - - - - con.
And the contained angles CEA, BEF are equal 70
Th. the angle ACE is equal to EBF - - 62
But the angle CBD is greater than EBF - 52
Th. the angle CBD is greater than ACE, or ACB. 56
By the ſame reaſon the angle ABG is greater than BAC
And the angles CBD, ABG are equal - - 70
Th. the angle CBD is greater than BAC - 56

72. *In triangles, oppoſite to the greater ſide, is the greater angle.*

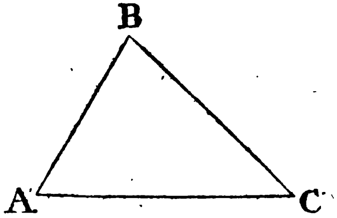


Let ABC be a triangle, having the ſide BC greater than the ſide BA, then the angle BAC is greater than the angle BCA.

For in BC take BD equal to BA by problem 3, and join AD.

Now the ſide BD is equal to the ſide BA - con.
Th. the angle BAD is equal to BDA - - 63
But the angle BDA is greater than BCA - 71
Th. the angle BAD is greater than BCA - 56
Th. the angle BAC is much greater then BCA - -

73. In triangles, oppoſite to the greater angle, is the greater ſide.

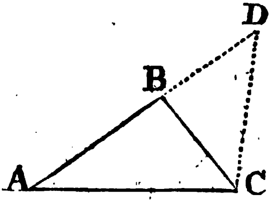


Let the triangle ABC, have the angle A greater than the angle C, then the ſide BC is greater than the ſide BA.

For if the ſide BC was leſs than the ſide BA, the angle
A would be leſs than the angle C - - - 72
But it is not ſo - - - - - theo.
Th. BC is not leſs than BA - - - - -
Again, if the ſide BC was equal to the ſide BA, the
angle A would be equal to the angle C - - 63
But thoſe angles are not equal - - - theo.
Th. BC is not equal to BA - - - - -
Conſequently, BC muſt be greater than BA - -

73. *Any*

74. *Any two ſides of a triangle taken together, are greater than the remaining ſide.*

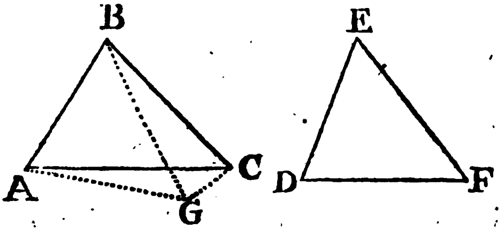


Let ABC be a triangle: then AB + BC is greater than CA, BC + CA is greater than AB, and CA + AB is greater than BC.

For produce AB to D, make BD equal to BC, and join CD.

Now the ſide BD is equal to the ſide BC - - con.
Th. the angle BCD is equal to BDC - - - 63
But the angle ACD is greater than BCD - - 52
Th. the angle ACD is greater than BDC - - 56
Th. the ſide AD is greater than the ſide CA - - 73
But AB + BC is equal to AD - - - 49
Th. AB + BC is greater than CA - - - 56
Alſo BC + CA is greater than AB } by the ſame reaſon.
And CA + AB is greater than BC }

75. *If triangles have two ſides equal to two ſides each to each, but the contained angles unequal; oppoſite to the greater angle is the greater baſe.*

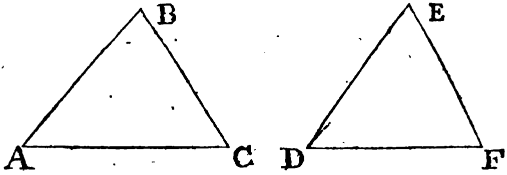


Let the triangles ABC, DEF have the ſides AB, BC equal to DE, EF each to each, but the angle ABC greater than the angle DEF; then the baſe AC is greater than the baſe DF.

For with AB, which is not the greater ſide, make the angle ABG equal to the angle DEF by problem 9, and the ſide BG equal to the ſide EF by problem 3, and join AG, GC.

Now the ſide BG is equal to the ſide BC - - 48
Th. the angle BCG is equal to BGC - - 63
But the angle BCG is greater than ACG - - 52
Th. the angle BGC is greater than ACG - - 56
Th. the angle AGC is much greater than ACG - - 52
Th. the ſide AC is greater than the ſide AG - 73
But AG and DF are equal - - - - 62
Th. the baſe AC is greater than the baſe DF - 56

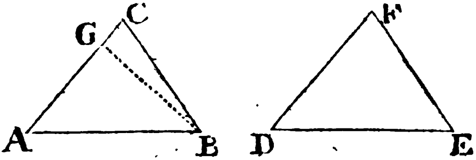
76. If triangles have two ſides equal to two ſides each to each, but the baſes unequal; oppoſite to the greater baſe is the greater angle.



Let the triangles ABC, DEF have the ſides AB, BC equal to DE, EF each to each, but the baſe AC greater than the baſe DF: then the angle B is greater than the angle E.

For if the angle B was equal to the angle E, the baſe
AC would be equal to the baſe DF - - 62
But the baſes are not equal - - - - theo.
Th. the angle B is not equal to the angle E - -
Again, if the angle B was leſs than the angle E, the
baſe AC would be leſs than the baſe DF - - 75
But AC is not leſs than DF - - - - theo.
Th. the angle B cannot be leſs than E - - -
Conſequently, the angle B muſt be greater than the
angle E - - - - - - - - -

77. *Triangles that have two angles equal to two angles each to each, and a ſide equal to a correſponding ſide; have alſo their remaining parts equal to one another each to each, namely, the parts oppoſite to equal things.*



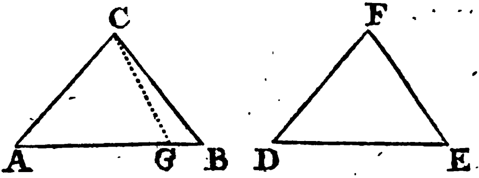
Let the triangles ABC, DEF have the angles ABC, CAB equal to DEF, FDE each to each, and the ſides AB, DE equal, that are between the equal angles: then the ſides AC, CB are equal to, DF, FE each to each, and the angle ACB is equal to DFE.

For if AC and DF are not equal, let AC be the greater, and in it take AG equal to DF by problem 3, and join GB.

Now the ſides BA, AG are equal to ED, DF each to each. - - - - - - con.
And the contained angles BAG, EDF are equal. theo.
Th. the angle ABG is equal to DEF - - 62
But the angle ABC is equal to DEF - - theo.
Th. the angles ABG, ABC are equal - - 48
Which is impoſſible - - - - - 52
Th. the ſides AC, DF muſt be equal - - -
Th. the ſides CB, FE are equal - - - 62
And the angle ACB is equal to DFE - - 62

77. Caſe 2.

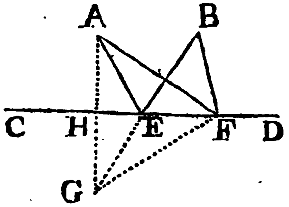
When the equal ſides are oppoſite to equal angles.



Let the triangles ABC, DEF have the angles ABC, CAB equal to DEF, FDE each to each, and the ſides AC, DF equal that are oppoſite to equal angles: then the ſides AB, BC are equal to DE, EF each to each, and the angle ACB is equal to DFE.

For if AB and DE are unequal, let AB be the greater, in which make AG equal to DE, and join CG.
Now the ſides CA, AG are equal to FD, DE each to each. - - - - - - - - - con.
And the contained angles CAG, FDE are equal. theo.
Th. the angle AGC is equal to DEF - - 62
But the angle ABC is equal to DEF - - - theo.
Th. the angles AGC, ABC are equal - - 48
Which is impoſſible - - - - - - 71
Th. the ſides AB, DE muſt be equal - - -
Th. the ſides BC, EF are equal - - - 62
And the angle ACB is equal to DFE - - 62

78. *If from two points on the ſame ſide of an infinite line, many pairs of lines be drawn to aſſumed points in it; the leaſt pair of all are thoſe which meet the ſaid line with equal angles.*

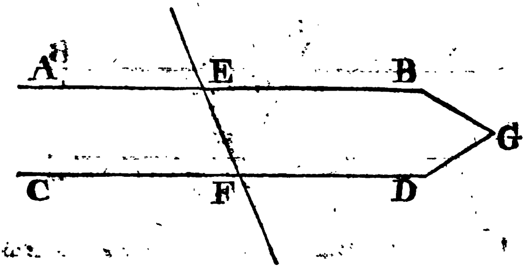


From the points A, B to the ſtrait line C D, let there be drawn AE, BE, making the angles AEC, BED equal to each other, and AF, BF at pleaſure; then AE + EB is leſs than AF + FB.

For produce BE to G, make EG equal to EA, and join AG, GF.

Now the angle AEH = BEF - - - theo.
And the angle GEH = BEF - - - - 70
Th. the angle AEH = GEH - - - - 48
And AE, EH are equal to GE, EH each to each. con.
Th. AH = GH, and the angle AHE = GHE - 62
Th. AF = GF. - - - - - - 62
Th. AF + FB = GF + FB - - - - 49
And AE + EB =(GE + EB =) GB - - - 49
But GB is leſs than GF + FB - - - 74
Th. AE + EB is leſs than AF + FB - - 56

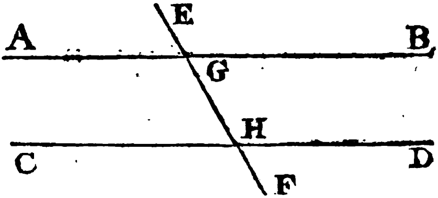
79. If a ſtrait line falling upon two ſtrait lines, makes the alternate angles equal to one another, theſe two lines are parallels.



Let the line EF falling upon the lines AB, CD, make the alternate angles AEF, EFD equal to each other: then the lines AB, CD are parallels.

For if not, let them be continued till they meet at G, and form the triangle EFG. - - - - - 59
Now the angle AEF is equal to EFG - - theo.
And the angle AEF is greater than EFG - - 71
Which is abſurd, for things cannot be equal and unequal. - - - - - - - - - -
Th. the lines AB, CD continued will not meet. - -
Th. AB, CD are parallels. - - - - 12

80. *If a ſtrait line falling upon two ſtrait lines, makes the outward angles equal to the inward and oppoſite angles on the ſame ſide, thoſe two lines are parallels.*



Let the line EF falling upon the lines AB, CD, make the angle EGB equal to the angle GHD; then AB, CD are parallels.

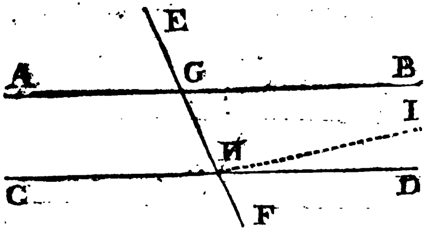
For the angle EGB is equal to AGH - - 70
And the angle EGB is equal to GHD - - theo.
Th. the angle AGH is equal to GHD - - 48
And they are alternate angles - - - - -
Th. AB, CD are parallels - - - 79

81. Corollary.

If a ſtrait line falling upon two ſtrait lines, makes the inward angles on the ſame ſide equal to two right angles, thoſe two lines are parallels.

For if the angle BGH + GHD = two right angles. theo.
Alſo the angle AGH + BGH = two right angles. 66
Th. the angle AGH + BGH = BGH + GHD. - 48
Th. the angle AGH - - - - = - - - - GHD. - 50
Th. AB, CD are parallels. - - - - 79

[illegible] A ſtrait line falling upon parallels, makes the alternate angles equal to one another.



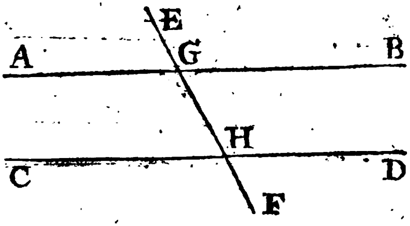
Let the line EF fall upon the parallels AB, CD at G and H; then the alternate angles AGH, GHD are equal.

For if not, make the angle GHI equal to AGH. p. 9.
Now the angles AGH, GHI are equal - - con.
And they are alternate angles - - - -
Th. the lines AB, HI are parallels - - - 79.
But the lines AB, CD are parallels - - theo.
Th. the lines HI, CD meeting in H, are both parallel to the line AB - - - - -
Which is impoſſible - - - - 57
Th. the angles AGH, GHD muſt be equal. - -

81. Cor.

83. Corollary.

A line falling upon parallels, makes the outward angles equal to the inward and oppoſite angles on the ſame ſide.



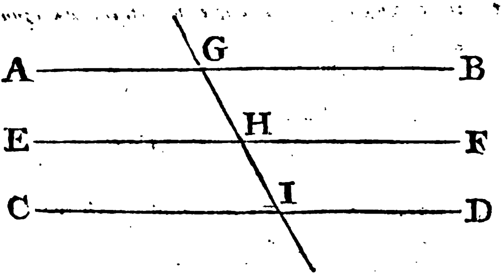
For the angle EGB is equal to AGH - - 70
And the angle GHD is equal to AGH - - 82
Th. the angle EGB is equal to GHD - - 48

84. Corollary.

A line falling upon parallels, makes the two internal angles on the ſame ſide equal to two right angles.

For the angle AGH + HGB = two right angles. - 69
Th. the angle GHD + HGB = two right angles - 56

85. *Lines that are parallel to the ſame line, are parallel to one another.*



Let each of the lines AB, CD be parallel to the line EF; then AB, CD are parallels.

For let the ſtrait line GHI fall upon them.

Now the angle AGI is equal to GHF - - 82

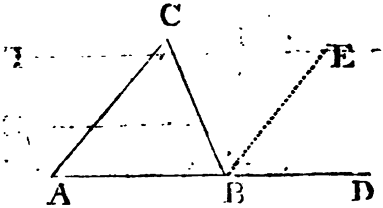
And the angle GID is equal to GHF - - 83

Th. the angle AGI is equal to GID - - 48

And they are alternate angles - - - -

Th. AB, CD are parallels - - - - 79

86. If any ſide of a triangle is produced, the outward angle will be equal to both the inward and remote angles; and the three angles of the triangle altogether, are equal to two right angles.



Let the triangle ABC have the ſide AB produced to D: then the angle CBD is equal to both the angles ACB, BAC, and the three angles of the triangle are equal to two right angles.

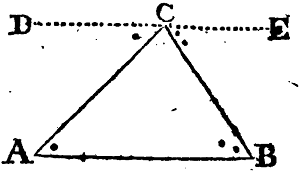
For draw BE parallel to AC - - - p. 10
Now the angle CBE - - - - - - = ACB - - 82
And the angle EBD - - - - - - = BAC - - 83
Th. the angle CBD - - - - - - = ACB+BAC - 49

Secondly,

The angle - - CBD+CBA - - - = ACB+BAC+CBA. 49
But the angle CBD+CBA - - - = two right angles. 67
Th. the angle ACB+BAC+CBA = two right angles. 48

Again,

The three angles of a plain triangle ABC *altogether are equal to two right angles.*



For thro' the vertex C, draw DE parallel to the baſe AB, by problem 10.

Now the angle ABC - - - - - - - = BCE . . 82
Th. the angle ABC+BCA - - - = ECA - - 49
Ag. the angle - - - - - - - CAB = ACD - - 82
Th. the angle ABC+BCA+CAB = ECA+ACD - 49
But two right angles - - - - - - - = ECA+ACD - 67
Th. the angle ABC+BCA+CAB = two right angles. 48

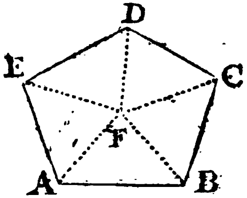
87. Corollary.

If two angles of one triangle be equal to two angles of another, the remaining angles are alſo equal.

88. Corollary.

In right angled triangles, the right angle is equal to the ſum of both the acute angles.

89. *All the angles of a figure (when they are all inward) are equal to twice as many right angles, excepting four, as the figure has ſides.*

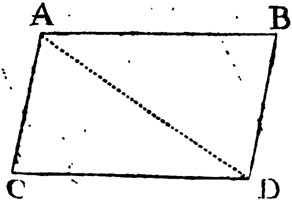


Let ABCDE be a right lined figure; then the ſum of all its angles A, B, C, D, E are equal to twice as many right angles, excepting four, as the figure has ſides.

For aſſume the point F within the figure, and join FA, FB, FC, &c.

Now the figure is divided into as many triangles as it has ſides - - - - - - con.
And each triangle contains two right angles - 86.
Therefore, all the triangles contain twice as many right angles as the figure has ſides - - -
But all the angles at the point F taken together, are equal to four right angles - - - - - 68.
And the remaining angles of the triangles compoſe the angles of the figure - - - - -
Conſequently, all the angles of the figure taken together, are equal to twice as many right angles, excepting four, as the figure has ſides - - - -

90. *If the ends of two equal and parallel lines be joined by lines not cutting one another, theſe lines will alſo be equal and parallel.*

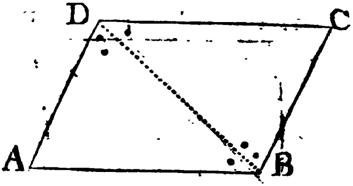


Let the equal and parallel lines AB, CD be joined by AC, BD: then the lines AC, BD are likewiſe equal and parallel.

For draw the diagonal AD.

Now the line AB is equal to the line CD. - theo.
Th. the ſides BA, AD are equal to CD, DA each to each.
And the contained angles BAD, CDA are equal. - 82
Th. the line AC is equal to the line BD - - 62
Alſo the angle ADB is equal to DAC - - 62
And they are alternate angles - - - -
Th. AC, BD are parallels - - - - 79

91. *The oppoſite ſides, and oppoſite angles of a parallelogram are equal, and a diagonal line divides it into equal parts.*



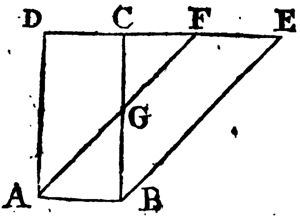
Let ABCD be a parallelogram, with a diagonal line BD; I ſay, its oppoſite ſides are equal, its oppoſite angles are equal, and the triangles BAD, DCB are equal.

For the angle BDA is equal to DBC - - 82
- - the angle ABD is equal to CDB - - 82
Th. the angles BDA, ABD = DBC, CDB each to each.
And the correſponding ſide BD is common - -
Th. the ſides BA, AD = DC, CB each to each - 77
And the angles BAD, DCB are equal - - 77
Th. the triangles BAD, DCB are equal - - 62

92. Corollary.

A parallelogram that has a right angle, is a rectangle. 23

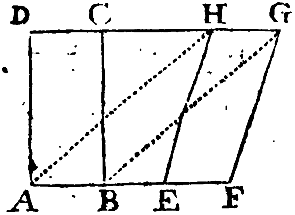
93. *Parallelograms upon the ſame baſe, and between the ſame parallels, are equal to one another.*



Let ABCD, ABEF be parallelograms upon the baſe AB, and between the parallels AB, DE: then ABCD is equal to ABEF.

For DC and FE are each equal to AB - - 91
Th. DC is equal to FE - - - - - 48
Th. the line DF is equal to the line CE - - 49
Ag. the ſide AD is equal to the ſide BC - - 91
Th. AD, DF are equal to BC, CE each to each - -
And the contained angles ADF, BCE are equal - 83
Th. the triangle ADF is equal to the triangle BCE 62
Th. the quadrangle ADCG is equal to BGFE - 50
Th. the parallelogram ABCD is equal to ABEF - 49

94. *Parallelograms upon equal baſes, and between the ſame parallels, are equal to one another.*



Let ABCD, EFGH be parallelograms upon equal baſes AB, EF, and between the parallels AF, DG: then ABCD, EFGH are equal.

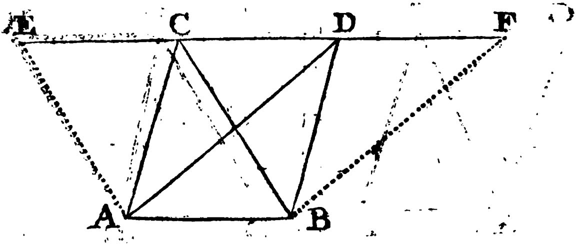
For join AH, BG.

Now AB is equal to EF - - - -	theo.
And HG is equal to EF - - - -	91
Th. the lines AB, HG are equal - - -	48
Th. the lines AH, BG are parallels - -	90
Th. ABGH is a parallelogram - - -	22
Th. ABGH is equal to ABCD - - -	93
Alſo ABGH is equal to EFGH - - -	93
Th. the parallelograms ABCD, EFGH are equal -	48

Remark,

All parallelograms of equal baſe and altitude, are equal.

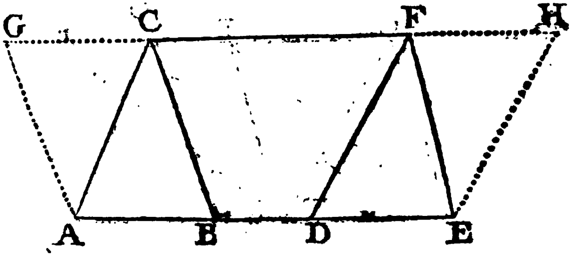
95. *Triangles upon the ſame baſe, and between the ſame parallels, are equal to one another.*



Let ABC, ABD be triangles upon the baſe AB, and between the parallels AB, CD: then the triangles ABC, ABD are equal.

For produce CD both ways to E and F, and draw AE, BF parallel to BC, AD - - - p. 10
Now ABCE, ABFD are parallelograms - - 22
And ABCE is equal to ABFD - - - 93
But the triangle ABC is half ABCE - - 91
And the triangle ABD is half ABFD - - 91
Th. the triangle ABC is equal to ABD - - 51

96. Triangles upon equal baſes, and between the ſame parallels, are equal to one another.



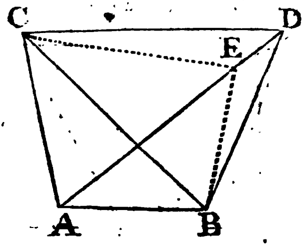
Let ABC, DEF be triangles upon equal baſes AB, DE, and between the parallels AE, CF: then the triangles ABC, DEF are equal.

For produce CF both ways to G and H, and draw
AG, EH parallel to BC, DF - - - - p. 16
Now ABCG, DEHF are parallelograms - - 22
And ABCG is equal to DEHF - - - 94
But the triangle ABC is half ABCG - - 91
And the triangle DEF is half DEHF - - 91
Th. the triangle ABC is equal to DEF - - 51

Remark,

All triangles of equal baſe and altitude, are equal.

97. *Equal triangles upon the ſame baſe, and on the ſame ſide, are between the ſame parallels.*

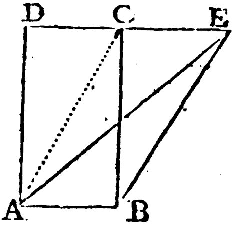


Let ABC, ABD be equal triangles ſtanding upon the baſe AB, and join CD: then the line CD is parallel to the line AB.

For if not, draw CE parallel to AB by problem 10, and join BE.

Now the triangle ABC is equal to ABE - - 95
And the triangle ABC is equal to ABD - - theo.
Th. the triangles ABE, ABD are equal - - 48
Which is impoſſible - - - - - - 52
Th. the line CE is not parallel to the line AB - -
Conſequently, CD muſt be parallel to AB - - -

98. *If a parallelogram and a triangle have the ſame baſe, and are between the ſame parallels, the parallelogram will be double to the triangle.*



Let the parallelogram ABCD, and the triangle ABE be conſtituted upon the baſe AB, and between the parallels AB, DE: then the parallelogram ABCD is double to the triangle ABE.

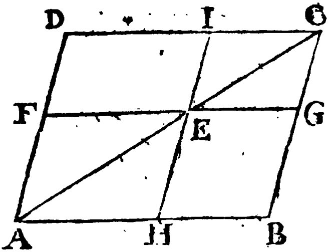
For join AC.

Now the triangles ABC, ABE are equal - - 95
And the triangle ABC is half ABCD - - 91
Th. the triangle ABE is equal to half ABCD - 56
Th. ABCD is double to the triangle ABE - - -

Remark,

A parallelogram is double to a triangle of an equal baſe and altitude.

97. *The complements in a parallelogram are equal.*

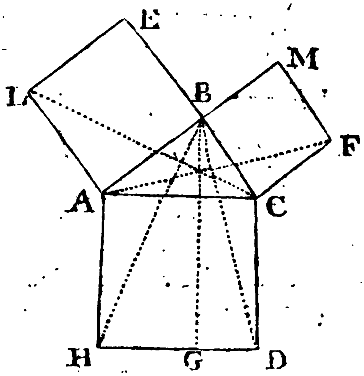


Let ABCD be a parallelogram; and thro' any point E in the diagonal line AC, let FG, HI be drawn parallel to its ſides: then the parallelograms DE, EB formed on each ſide of the diagonal are called the complements of thoſe FH, IG, which are formed about it: I ſay, DE is equal to EB.

For AFEH, EICG are parallelograms - - 22
Th. the triangle AFE - - - - = AHE - - 91
And the triangle EIC - - - - = EGC - - 91
Th. - - - - - - - AFE + EIC = AHE + EGC - 49
Ag. the triangle - - - - ADC = ABC - - 91
Th. the complement - - - DE = EB - - 50

98. *In*

100. *In right angled triangles, the ſquare of the ſide oppoſite to the right angle, is equal to both the ſquares of the ſides containing it.*

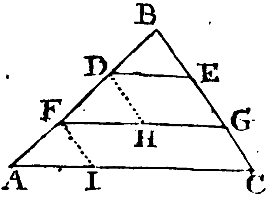


Let the triangle ABC have a right angle at B, then $AC^2 = AB^2 + BC^2$.

For upon AB, BC, CA deſcribe the ſquares AE, CM, CH by problem 11; draw BG parallel to AH or CD by problem 10, and join AF, CL.

Now the angle LAB is equal to CAH - - 54
Th. the angle LAC is equal to BAH - - 49
And LA, AC are equal to BA, AH each to each - 24
Th. the triangles LAC, BAH are equal - - 62
Again, EBC is a ſtrait line - - - - 69
Th. the ſquare AE is double to the triangle LAC. 98
Alſo, the parall. AG is double to the triangle BAH. 98
Th. the parallelog. AG is equal to the ſquare AE. 51
By the ſame reaſon GC is equal to the ſquare CM - -
Th. $AG + GC = AE + CM$ - - - - 49
Or - - - $AC^2 = AB^2 + BC^2$ - - - - 61

101. *If ſtrait lines drawn parallel to the baſe of a triangle divide one of its ſides into equal parts, they will alſo divide the other ſide into equal parts.*



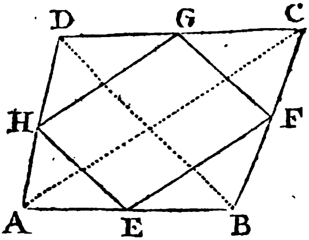
Let ABC be a triangle, whoſe ſide AB is divided into equal parts by the lines DE, FG drawn parallel to the baſe AC; then the ſide BC will likewiſe be divided into equal parts by the ſaid lines.

For let DH, FI be drawn parallel to the ſide BC. p. 10
Now the angles DBE, FDH, AFI are equal - - 83
Alſo, the angles BDE, DFH, FAI are equal - 83
And the correſponding ſides BD, DF, FA are equal. theo.
Th. the ſides BE, DH, FI are equal - - 77
But EG, GC are equal to DH, FI - - - 91
Th. the parts BE, EG, GC are equal - - 48

102. Corollary.

A right line that biſects two ſides of a triangle, muſt be parallel to the remaining ſide.

103. *If the middle points in the ſides of any quadrangle are joined by four ſtrait lines, they will form a parallelogram.*



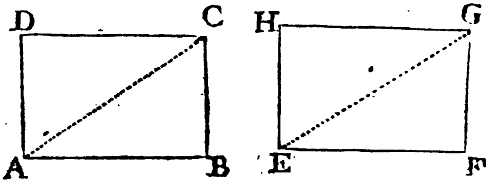
Let there be a quadrangle ABCD whoſe ſides are biſected in the points E, F, G, H, and joined by the ſtrait lines EF, FG, GH, HE; then the figure EFGH is a parallelogram.

For draw the diagonals AC, BD.

Now the ſides AB, BC are biſected in E and F. theo.
Th. EF is parallel to AC - - - - 102
By the ſame reaſon HG is parallel to AC - - -
Th. the lines EF, HG are parallel - - 85
By the ſame reaſon EH, FG are parallel - - -
Th. EFGH is a parallelogram - - - 22

BOOK II.

104. *The rectangles contained by equal lines, are equal to one another.*



Let ABCD, EFGH be rectangles, having the ſides AB, BC equal to EF, FG each to each; then the rectangles ABCD, EFGH are equal, or AB × BC = EF × FG.

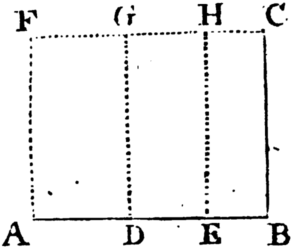
For draw the diagonals AC, EG.

Now the ſides AB, BC are equal to EF, FG each to each - - - - - - - - theo.
And the contained angles ABC, EFG are equal - 54
Th. the triangles ABC, EFG are equal - - 62
By the ſame reaſon, the triangles ADC, EHG are equal.
Th. ABCD is equal to EFGH - - - 49
Or AB × BC = EF × FG - - - - 61

105. Corollary.

The ſquares of equal lines are equal to one another.

106. *The rectangle contained by a whole and a divided line, is equal to all the rectangles contained by the ſeveral parts of the divided line, and the whole line.*

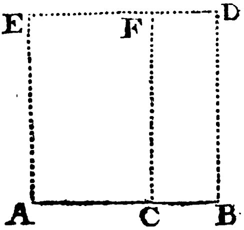


Take the ſtrait lines AB, BC at pleaſure, and divide AB any how at D and E; I ſay, $AB \times BC = AD \times BC + DE \times BC + EB \times BC$.

For erect BC at right angles to AB by problem 6, draw CF parallel to BA, and AF, DG, EH parallel to BC by problem 10.

Now ABC is a right angle - - - con.
Th. AC, AG, GE, EC are rectangles - - 92
And AC is equal to AG + GE + EC - - - 52
Or $AB \times BC = AD \times DG + DE \times EH + EB \times BC$ 61
But BC, DG, EH are equal - - - - 91
Th. $AB \times BC = AD \times BC + DE \times BC + EB \times BC$ 56

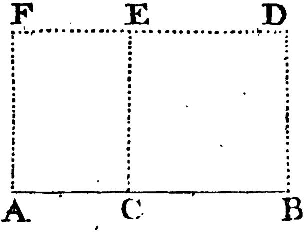
107. *If a ſtrait line is any how divided, the rectangles contained by each of the parts and the whole line, will be equal to the ſquare of the ſaid line.*



Let the ſtrait line AB be divided at pleaſure in C; then $AC \times AB + CB \times AB = AB^2$.

For upon AB deſcribe the ſquare ABDE by problem
11, and draw CF parallel to AE or BD - p. 10
Now AF, FB are rectangles - - - 92
And $AF + FB$ is equal to AD - - - 52
Or $AC \times AE + CB \times BD = AB^2$ - - - 61
But AE, AB, BD are equal - - - - 24
Th. $AC \times AB + CB \times AB = AB^2$ - - - 56

108. *If a ſtrait line is divided into any two parts, the rectangle contained by the whole line and either part, will be equal to the rectangle contained by the parts with the ſquare of the part taken.*

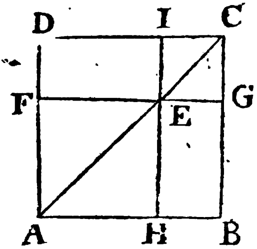


Let the ſtrait line AB be any how divided in C, and take the part CB; then $AB \times CB = AC \times CB + CB^2$.

For upon CB deſcribe the ſquare CBDE, produce DE to F, and draw AF parallel to CE or BD.

Now AD, AE are rectangles - - -	92
And AD is equal to AE + EB - - -	52
Or $AB \times BD = AC \times CE + CB \times BD$ - - -	61
But CE, CB, BD are equal - - - -	24
Th. $AB \times CB = AC \times CB + CB^2$ - - -	56

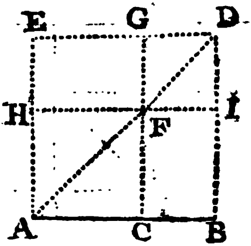
109. *The parallelograms formed about the diagonal of a ſquare, by lines drawn parallel to its ſides, are likewiſe ſquares.*



Let ABCD be a ſquare, and through any point E in the diagonal line AC, draw FG, HI parallel to AB, BC; then FH, IG are the ſquares of AH, HB.

For the angle ACB is equal to BAC - - 63
And the angle ACB is equal to AEH - - 83
Th. the angles BAC, AEH are equal - - 48
Th. the ſides AH, HE are equal - - - 65
But the ſides EF, FA are equal to AH, HE - 91
Th. AH, HE, EF, FA are equal - - - 48
Again, the angles at A, H, E, F are right ones, for they are equal to the angles of the ſquare ABCD. - 91
Th. AHEF is the ſquare of AH - - - 24
And EGCI is the ſquare of IC or HB, by the ſame reaſon

110. *If a ſtrait line is divided into any two parts, the ſquare of the whole line will be equal to the ſquares of the parts, with twice a rectangle contained by the parts.*

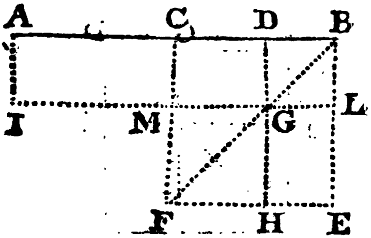


Let the ſtrait line AB be divided at pleaſure in C; I ſay, $AB^2 = AC^2 + CB^2 + 2AC \times CB$.

For upon AB deſcribe the ſquare ABDE, join AD, and draw CFG, HFI parallel to AE, AB.

Now $ABDE = HC + GI + EF + FB$ - - - - 92
And - - $EF = FB$ - - - - - - 99
Th. $ABDE = HC + GI + 2EF$ - - - - 56
Or - - - $AB^2 = HC + GI + 2HF \times FG$ - - 61
But HC, GI are the ſquares of AC, CB - - 109
Th. - - $AB^2 = AC^2 + CB^2 + 2AC \times CB$ - - 56

111. *If a ſtrait line is equally and unequally divided, the rectangle contained by the unequal parts with a ſquare of the intermediate diſtance, will be equal to the ſquare of half the line.*



Let the ſtrait line AB be divided equally at C, and any how at D: I ſay, $AD \times DB + CD^2 = CB^2$.

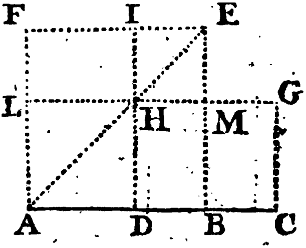
For upon CB deſcribe the ſquare CBEF, join BF, draw AI, DGH parallel to BE, and IGL parallel to AB.

Now $CG = GE$ - - - 99
Th. $CL = DE$ - - - 49
But $AM = CL$ - - - 94
Th. $AM = DE$ - - - 56
Th. $AG = DE + CG$ - - 49
Th. $AG + MH = DE + CG + MH$ - - 49
Or $AD \times DG + MH = CB^2$ - - 61
But MH, DL are the ſquares of CD, DB - - 109
Th. $AD \times DB + CD^2 = CB^2$ - - - 56

111*. Other-

*111. Otherwiſe,

The rectangle contained by the ſum and difference of two lines, is equal to the difference of their ſquares.

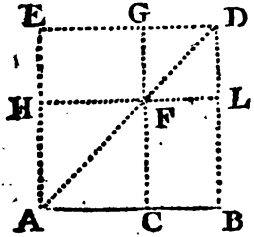


Let AB, BC be unequal lines in the ſame direction, whereof AB is the greater, take BD equal to BC, then AC is their ſum, and AD is their difference: I ſay, $AC \times AD = AB^2 - BC^2$.

For upon AB deſcribe the ſquare ABEF, join AE, draw CG, DHI parallel to BE, and LHG parallel to AC.

Now $BG = BH$ - - - 94
And $FH = BH$ - - - 99
Th. $BG = FH$ - - - 48
Th. $BG + LB = FH + LB$ - - - 49
That is $AG = BF - IM$ - - - -
Or $AC \times AL = AB^2 - IM$ - - - 61
But LD, IM are the ſquares of AD, DB or BC - 109
Th. $AC \times AD = AB^2 - BC^2$ - - - 56

112. If a ſtrait line is divided into any two parts, the ſquares of the whole line and either part, will be equal to twice a rectangle contained by the ſaid line and part taken with the ſquare of the other part.



Let the right line AB be divided any how in C, and take the part BC: I ſay, $AB^2 + BC^2 = 2AB \times BC + AC^2$.

For upon AB deſcribe the ſquare ABDE, join AD, draw CFG parallel to AE, and HFL parallel to AB.

Now $EF = FB$ - - - - 99

Th. $EB = 2EF + GL + HC$ - - 52

Th. $EB + GL = 2EF + 2GL + HC$ - - 49

Th. $EB + GL = 2EL + HC$ - - - - 56

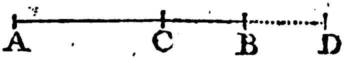
Or $AB^2 + GL = 2ED \times DL + HC$ - - - 61

But GL, HC are the ſquares of BC, CA - - 109

Th. $AB^2 + BC^2 = 2AB \times BC + AC^2$ - - 56

113. If

112. *If a line is divided into any two parts, four times the rectangle contained by the whole line and either part with the ſquare of the other part, will be equal to the ſquare of the ſum of the ſaid line and part taken.*



Let the ſtrait line AB be divided at pleaſure in C, and take the part BC: I ſay, $4AB \times BC + AC^2 = \overline{AB+BC}^2$.

For produce AB to D, and make BD equal to BC.

Now $AB^2 + BD^2 + 2AB \times BD$ - - - $= AD^2$ - 110

Th. $AB^2 + BC^2 + 2AB \times BC$ - - - $= AD^2$ - 56

Ag. $2AB \times BC + AC^2$ - - - - - - - - $= AB^2 + BC^2$ . . 112

Th. $2AB \times BC + AC^2 + 2AB \times BC = AD^2$ - - 56

Or $4AB \times BC + AC^2$ - - - - - - - $= AD^2$ - - 56

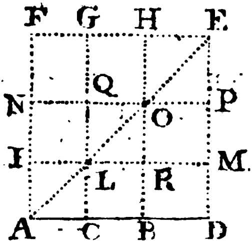
But $AB + BC$ - - - - - - - - - - - - - $= AD$ - con.

Th. $4AB \times BC + AC^2$ - - - - - - - $= \overline{AB+BC}^2$. 56

113. Other

113. Otherwiſe,

If a line is divided equally, and another be added to it, the ſquare of the ſum, will be equal to four times a rectangle contained by the half line and remainder, with the ſquare of the added line.

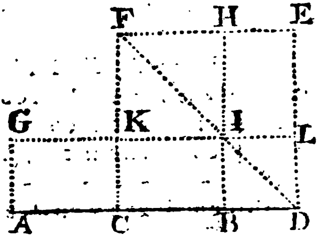


Let the ſtrait line AB be equally divided in C, and continued to D; then $AD^2 = 4AC \times CD + BD^2$.

For upon AD deſcribe the ſquare ADEF, join AE, draw CG, BH parallel to AF, and ILM, NOP parallel to AD.

Now NB, IC, QR, HP are the ſquares of AB, AC, CB, BD. - - - - - - - - 109
And AC, CB are equal - - - - theo.
Th. IC, QR, BL, LN, are equal - - - 105
Ag. NG = GO, and OM = MB - - - 94
But GO is equal to OM - - - - 99
Th. NG, GO, OM, MB are equal - - 48
Th. IG, GR, LD, and IC + OM are equal - 49
Th. FO + ND = 4LD - - - - 49
Th. FO + ND + HP = 4LD + HP - - - 49
Or - - $AD^2 = 4CL \times CD + BD^2$ - - - 61
Th. $AD^2 = 4AC \times CD + BD^2$ - - - 56

114. *If a line is divided equally, and another be added to it, the rectangle contained by the ſum and added line with the ſquare of half the line, will be equal to the ſquare of the line compoſed of the half and added line.*

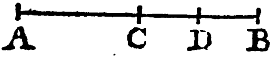


If the ſtrait line AB is divided equally in C, and produced to D; then $AD \times DB + CB^2 = CD^2$.

For let CDEF be the ſquare of CD, join DF, draw AG, BH parallel to CF, and through I draw GL parallel to AD.

Now AK - - - - - - - - = CI - - - 94
And IE - - - - - - - - = CI - - - 99
Th. AK - - - - - - - - = IE - - - 48
Th. $AK + CL$ - - - - = $IE + CL$ - - 49
Or AL - - - - - - - - = $CL + IE$ - - -
Th. $AL + KH$ - - - - = $CL + IE + KH$ - - 49
Or $AD \times DL + KH$ - - = CD^2 - - - 61
But KH, BL are the ſquares of CB, BD - - 109
Th. $AD \times DB + CB^2$ - - = CD^2 - - - 56

115. *If a line is equally and unequally divided, the ſquares of the unequal parts, will be double to the ſquares of the half line and intercepted part.*



Let the ſtrait line AB be equally divided in C, and any how in D; then $AD^2 + DB^2 = 2AC^2 + 2CD^2$ - -

For AD^2 - - - $= AC^2 + CD^2 + 2AC \times CD$ - - - 110

Th. $AD^2 + DB^2 = AC^2 + CD^2 + 2AC \times CD + DB^2$. 49

But BC - - - $= AC$ - - - - theo.

Th. $AD^2 + DB^2 = BC^2 + CD^2 + 2BC \times CD + DB^2$. 56

Ag. $BC^2 + CD^2 =$ - - - - - - - $2BC \times CD + DB^2$. 112

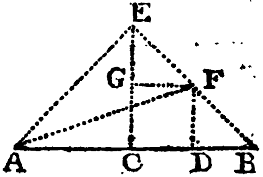
Th. $AD^2 + DB^2 = BC^2 + CD^2 + BC^2 + CD^2$ - 56

Or $AD^2 + DB^2 = 2BC^2 + 2CB^2$ - - - 52

Th. $AD^2 + DB^2 = 2AC^2 + 2CD^2$ - - - 56

115. Otherwiſe.

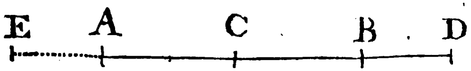
If the line AB *is equally divided in* C, *and at pleaſure in* D: *I ſay,* $AD^2 + DB^2 = 2AC^2 + 2CD^2$.



For erect CE at right angles to AB, and make it equal to AC, join AE, EB, draw DF, FG parallel to CE, BA, and join AF.

Now the angle AEC = CAE, and BEC = CBE - - 63
Th. each is half a right angle - - - - 88
Th. AEB is a right angle - - - - - 52
Alſo EGF, FDB are right angles, for each is equal to the right angle ECB - - - - - - 83
And the angles EFG, DFB are half right angles, being equal to the angles EBC, CEB - - 83
Th. EG is equal to GF, and FD is equal to DB. - 65
Ag. CD is equal to GF - - - - 91
Th. $AC^2 = CE^2$, $EG^2 = GF^2 = CD^2$, and $FD^2 = DB^2$. 105
Th. $AE^2 = 2AC^2$, $EF^2 = 2GF^2$, and $AF^2 = AE^2 + EF^2$. 100
Th. $AF^2 = 2AC^2 + 2GF^2$ - - - - 56
But $AF^2 = AD^2 + DF^2$ - - - - 100
Th. $AD^2 + DF^2 = 2AC^2 + 2GF^2$ - - - 48
Th. $AD^2 + DB^2 = 2AC^2 + 2CD^2$ - - - 56

116. *If a line is equally divided, and another be added to it, the ſquares of the ſum and added line, will be double to the ſquares of half the line, and the line compoſed of the half and added line.*

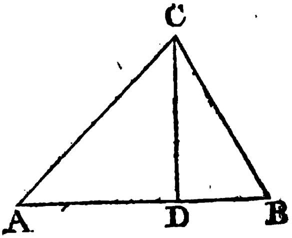


Let the ſtrait line AB be divided equally in C, and continued to D; then $AD^2 + BD^2 = 2AC + 2CD^2$.

For produce DA to E, and make AE equal to BD.

Now EA is equal to BD - - - - con.
And AC is equal to CB - - - - theo.
Th. EC is equal to CD - - - - 49
And EB is equal to AD - - - - 49
Therefore, the line ED is equally divided in C, and unequally in B - - - - - - -
Th. $EB^2 + BD^2 = 2CD^2 + 2CB^2$ - - - 115
But the ſquares of EB and AD are equal - - 105
Th. $AD^2 + BD^2 = 2CD^2 + 2CB^2$ - - - 56
Or $AD^2 + BD^2 = 2AC^2 + 2CD^2$ - - - 56

117. *In triangles, the ſquares of the baſe and either ſide, are equal to twice a rectangle contained by the baſe and adjacent ſegment made by a perpendicular, with the ſquare of the other ſide.*



Let the triangle ABC, have a perpendicular CD falling from the vertex to the baſe, and take the ſide BC: then $AB^2 + BC^2 = 2AB \times BD + AC^2$.

For $AB^2 + BD^2$ - - - $= 2AB \times BD + AD^2$ - 112

Th. $AB^2 + BD^2 + DC^2 = 2AB \times BD + AD^2 + DC^2$. 49

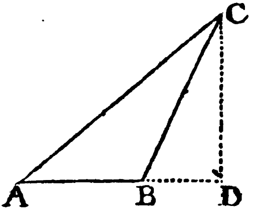
But BC^2 - - - - - $= BD^2 + DC^2$ - - - 100

Th. $AB^2 + BC^2$ - - - $= 2AB \times BD + AD^2 + DC^2$. 56

Ag. AC^2 - - - - - - - $=$ - - - - - - - - $AD^2 + DC^2$. 100

Th. $AB^2 + BC^2$ - - - - $= 2AB \times BD + AC^2$ - - 56

118. *In obtuſe angled triangles, the ſquare of the ſide oppoſite to the obtuſe angle, is equal to the ſquares of the ſides about it, with twice a rectangle contained by either of thoſe ſides and its continuation to a perpendicular from the extremity of the other ſide.*



Let the triangle ABC have an obtuſe angle at B, and let CD be perpendicular to the continuation of AB; then $AC^2 = AB^2 + BC^2 + 2AB \times BD$.

For $AC^2 = AD^2 + DC^2$ - - - - - 100

And $AD^2 = AB^2 + BD^2 + 2AB \times BD$ - - 110

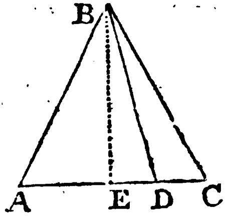
Th. $AC^2 = AB^2 + BD^2 + 2AB \times BD + DC^2$. - 56

Or $AC^2 = AB^2 + BD^2 + DC^2 + 2AB \times BD$ - - 52

But $BC^2 = \text{- - -}\ BD^2 + DC^2$ - - - 100

Th. $AC^2 = AB^2 + BC^2 + 2AB \times BD$ - - - 56

119. *In iſoſceles triangles, the ſquare of the ſide is equal to the ſquare of a line joining the vertex and any point in the baſe, with a rectangle contained by the parts of the baſe.*



Let the triangle ABC have the ſide AB equal to the ſide BC, and join the vertex to any point D in the baſe, then $AB^2 = BD^2 + AD \times DC$.

For divide AC equally in E by problem 5, and join BE.

Now EA, AB, BE = EC, CB, BE each to each. con.

Th. the angles AEB, BEC are equal - - 66

Th. AEB, BEC are right angles - - - 9

Th. $AB^2 = BE^2 + EA^2$ - - - - 100

But $AD \times DC + ED^2 = EA^2$ - - - 111

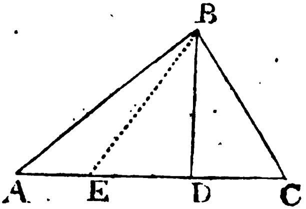
Th. $AB^2 = BE^2 + AD \times DC + ED^2$ - - 56

Or $AB^2 = BE^2 + ED^2 + AD \times DC$ - - 52

Ag. $BD^2 = BE^2 + ED^2$ - - - - 100

Th. $AB^2 = BD^2 + AD \times DC$ - - - 56

120. *The difference of the ſquares of two ſides of a triangle, is equal to the difference of the ſquares of the ſegments of the baſe made by a perpendicular.*



Let the triangle ABC have the ſide AB greater than the ſide BC, and let BD be perpendicular to the baſe AC: then $AB^2 - BC^2 = AD^2 - DC^2$.

For $AB^2 = AD^2 + DB^2$ - - - 100
And $BC^2 = DC^2 + DB^2$ - - - 100
Th. $AB^2 - BC^2 = AD^2 - DC^2$ - - - 50

121. Corollary.

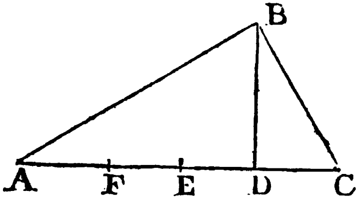
The difference of the ſquares of two ſides of a triangle, is equal to a rectangle contained by the baſe and difference of the ſegments made by a perpendicular.

For make DE equal to DC.

Now $AC = AD + DC$, and $AE = AD - DC$, which are the ſum and difference of the ſegments.

Th. $AC \times AE = AD^2 - DC^2$ - - - 111
Th. $AB^2 - BC^2 = AC \times AE$ - - - 48

122. *The rectangle contained by the ſum and difference of the ſides of a triangle, will be double to the rectangle contained by the baſe and diſtance of the perpendicular from the middle of the baſe.*



Let ABC be a triangle with a perpendicular BD falling upon the baſe AC, whoſe middle point is E: then $\overline{AB+BC} \times \overline{AB-BC} = 2AC \times ED$.

For make AF equal to DC, whence $AD-DC=FD$. 50

Now $\overline{AB+BC} \times \overline{AB-BC} = AB^2-BC^2$ - - 111*

And AD^2-DC^2 - - - - $= AB^2-BC^2$ - - 120

Th. $\overline{AB+BC} \times \overline{AB-BC} = AD^2-DC^2$ - - 48

But $AC \times FD$ - - - - - $= AD^2-DC^2$ - - 111*

Th. $\overline{AB+BC} \times \overline{AB-BC} = AC \times FD$ - - 48

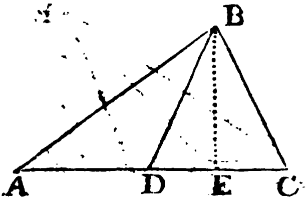
Ag. FE - - - - - - - - - - $= ED$ - - - 50

Th. FD - - - - - - - - - - $= 2ED$ - - - 49

Th. $\overline{AB+BC} \times \overline{AB-BC} = AC \times 2ED$ - - 56

Or $\overline{AB+BC} \times \overline{AB-BC} = 2AC \times ED$.

123. *If the baſe of a triangle is biſected by a line drawn from the vertex, the ſquares of the ſides will be double to the ſquares of the ſaid line and half the baſe.*



Let the triangle ABC, have its baſe AC equally divided by the line BD: then $AB^2+BC^2=2BD^2+2DC^2$.

For upon AC let fall the perpendicular BE - p. 7

Now $AB^2 - - - = AE^2 + EB^2$ - - - 100

And $BC^2 - - - = EC^2 + EB^2$ - - - 100

Th. $AB^2+BC^2 = AE^2 + EC^2 + 2EB^2$ - - 49

But $AE^2+EC^2 = 2DC^2 + 2DE^2$ - - 115

Th. $AB^2+BC^2 = 2DC^2 + 2DE + 2EB^2$ - - 56

Ag. $BD^2 - - - = - - - - DE^2 + EB^2$ - - 100

Th. $2BD^2 - - - = - - - - 2DE^2 + 2EB^2$ - - 49

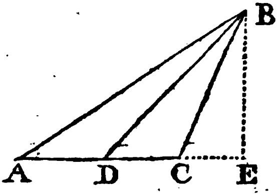
Th. $AB^2+BC^2 = 2DC^2 + 2BD^2$ - - - 56

Or $AB^2+BC^2 = 2BD + 2DC^2$ - - - -

Caſe 2.

123. Caſe 2.

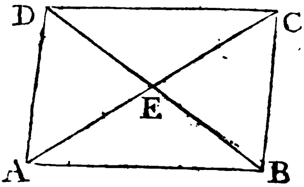
When the triangle ABC *has an obtuſe angle at the Baſe, I ſay as before,* $AB^2 + BC^2 = 2BD^2 + 2DC^2$.



For let BCA be the obtuſe angle, produce AC to E, and let BE fall at right angles to AE. - - p. 7.

Now AB^2 - - - $= AE^2 + EB^2$ - - - 100
And BC^2 - - - $= CE^2 + EB^2$ - - - 100
Th. $AB^2 + BC^2 = AE^2 + CE^2 + 2EB^2$ - - 49
But $AE^2 + CE^2 = 2DC^2 + 2DE^2$ - - 116
Th. $AB^2 + BC^2 = 2DC^2 + 2DE^2 + 2EB^2$. - - 56
Ag. BD^2 - - - = - - - - $DE^2 + EB^2$ - - 100
Th. $2BD^2$ - - - = - - - - $2DE^2 + 2EB^2$ - - 49
Th. $AB^2 + BC^2 = 2DC^2 + 2BD^2$ - - 56
Or $AB^2 + BC^2 = 2BD^2 + 2DC^2$.

124. *In parallelograms, the diagonal lines cut each other into equal parts, and their ſquares both together, are equal to the ſquares of the four ſides taken together.*



Let ABCD be a parallelogram whoſe diagonals AC, BD cut each other in E: then AE is equal to EC, BE is equal to ED, and $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$.

For the angles ABE, EAB are equal to CDE, ECD each to each. - - - - - - 82

And the correſponding ſides AB, CD are equal. 91

Th. AE is equal to EC, and BE is equal to ED - 77

Ag. $2AE^2+2EB^2=AB^2+BC^2$ - - - 123

Alſo $2AE^2+2ED^2=CD^2+DA^2$ - - 123

And - - - - $ED^2=EB^2$ - - - - 105

Th. - - - - $2ED^2=2EB^2$ - - - - 49

Th. $2AE^2+2EB^2=CD^2+DA^2$ - - - 56

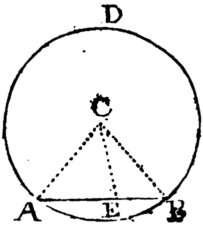
Th. $4AE^2+4EB=AB^2+BC^2+CD^2+DA^2$ - 49

But $4AE^2$ - - - - $=AC^2$, and $4EB^2=BD^2$. - 110

Th. $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$. - 56

BOOK III.

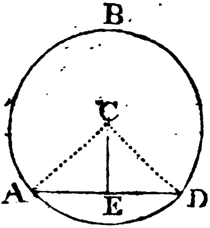
125. *A ſtrait line that joins any two points in the circumference of a circle, will fall within the circle.*



Let A and B be points in the circumference of a circle ADB, and join AB; then the ſtrait line AB is within the circle.

For find C the center of the circle by problem 32, in AB aſſume the point E, and join CE, CA, CB.
Now the ſide CA is equal to the ſide CB - - 30
Th. the angle CBA is equal to CAB - - - 63
But the angle CEB is greater than CAB - - 71
Th. the angle CEB is greater than CBA - - 56
Th. the line CB is greater than the line CE - 73
Th. the point E is within the circle - - 30
Th. the line AB is within the circle - - - -

126. A line drawn from the center of a circle, to the middle of a line terminated by the circumference, will meet the ſame at right angles.

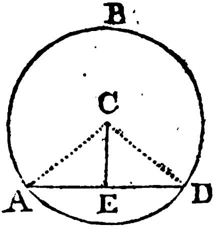


In a circle ABD, whoſe center is C, let the line AD be equally divided by the line CE: then AEC, CED are right angles.

For join AC, CD.

Now AE is equal to ED - - - - theq.
- - - AC is equal to CD - - - - 30
And CE is common - - - - - -
Th. EA, AC, CE are equal to ED, DC, CE each to each.
Th. the angles AEC, CED are equal - - 66
Th. AEC, CED are right angles - - - 9

127. *A line drawn from the center of a circle, at right angles to a line terminated by the circumference, will divide it equally.*

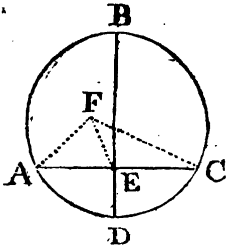


In the circle ABD, whoſe center is C, let the line CE be drawn at right angles to the line AD: then the ſegments AE, ED are equal.

For join AC, CD.

Now CD is equal to CA - - - - - 30
Th. the angles DAC, CDA are equal - - 63
Alſo the angles AEC, CED are equal - - 54
Th. the angles EAC, CEA are equal to EDC, CED each to each - - - - - - - - -
And the corresponding ſide CE is common.
Th. AE is equal to ED - - - - - 77

128. *If a line terminated by a circle, is divided equally by a line cutting it at right angles, the cutting line will paſs through the center of the circle.*

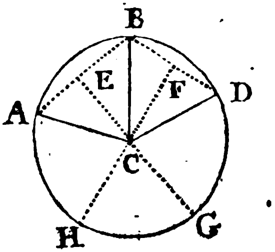


In the circle ABCD, let the line AC be equally divided in E, by the line BD cutting it at right angles; then BD will paſs thro' the center of the circle.

For if not, let F be the center, and join FA, FE, FC.

Now AE is equal to EC - - - - theo.
- - - FA is equal to FC - - - - 30
And EF is common - - - - - -
Th. EA, AF, FE = EC, CF, FE each to each.
Th. the angles AEF, FEC are equal - - 66
Th. each is a right angle - - - - 9
Th. the angle AEF is equal to AEB - - 54
Which is impoſſible - - - - - - 52
Th. BD muſt paſs thro' the center of the circle - -

129. *If from a certain point within a circle, three equal ſtrait lines be drawn to the circumference, that point is the center of the circle.*

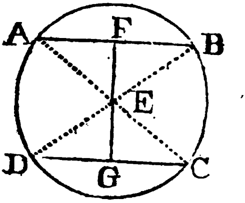


Let ABD be a circle in which three equal ſtrait lines CA, CB, CD are drawn from the point C to the circumference: then C is the centre of the circle.

For join AB, BD, biſect them in E and F by problem 5, draw EC, FC, and continue them to G and H.

Now CA is equal to CB - - - - - theo.
- - - AE is equal to EB - - - - - con.
And CE is common - - - - - - -
Th. the angles AEC, CEB are equal - - - 66.
Th. AEC, CEB are right angles - - - - 9
Th. the line ECG paſſes thro' the center - 128
By the ſame reaſon, the line FCH paſſes thro' the center.
Th. the point C muſt be the center of the circle.

130. *In a circle, equal ſtrait lines terminated by the circumference, are equally diſtant from the center.*



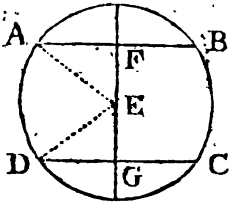
In the circle ABCD, whoſe center is E, let the ſtrait lines AB, DC be equal; then the perpendiculars EF, EG falling upon them, from the center, will be likewiſe equal.

For join EA, EB, EC, ED.

Now EA, EB, EC, ED are equal - - - 30
And AB is equal to DC - - - - theo.
Th. AE, EB, BA are equal to DE, EC, CD each to each. - - - - - - above
Th. the angle BAE = CDE - - - 66
Alſo the angle EFA = EGD - - - - 54
Th. the angles FAE, EFA are equal to GDE, EGD each to each. - - - - - - - above
And the correſponding ſides EA, ED are equal - 30
Th. EF and EG are equal - - - - 77

131. In

131. *In a circle, ſtrait lines equally diſtant from the center, and terminated by the circumference, are equal to one another.*



In the circle ABCD, let the ſtrait lines AB, DC be equally diſtant from the center E, namely, let the perpendiculars EF, EG falling upon them be equal: then AB, DC are equal.

For join AE, ED.

Now EA^2 - - - = $AF^2 + FE^2$ - - - 100

And ED^2 - - - = $DG^2 + GE^2$ - - - 100

But EA^2 - - - = ED^2 - - - - 105

Th. $AE^2 + FE^2 = DG^2 + GE^2$ - - - 48

Again, FE^2 = - - - - GE^2 - - - 105

Th. AE^2 - - - = DG^2 - - - - 50

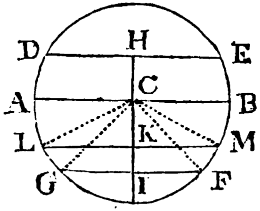
Th. AF - - - = DG - - - - 105

But AF - - - = FB, and DG = GC - - - 127

Th. AF, FB, DG, GC are equal - - - 48

Th. AB is equal to DC - - - - 49

132. *In circles, the greateſt line is a diameter, and of other lines the nearer to the center is the greater.*

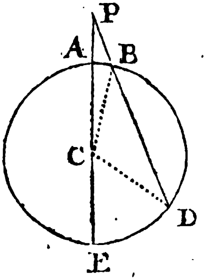


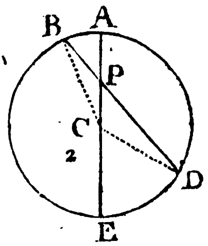
In the circle DEFG, whoſe center is C, and diameter AB, let the line DE be drawn nearer to the center than the line GF; then AB is greater than DE, and DE is greater than GF.

For upon DE, GF let fall the perpendiculars CH, CI, make CK equal to CH, through K draw LM at right angles to CI, and join CM, CF, CG, CL.

Now AC, CB, CM, CF, CG, CL are equal - - 30
Th. AB is equal to LC + CM - - 49
But LC + CM is greater than LM - - 74
Th. AB is greater than LM - - - 56
Ag. DE is equal to LM - - - - 131
Th. AB is greater than DE - - - - 56
Secondly, the angle LCM is greater than GCF 52
Th. LM is greater than GF - - - 75
Th. DE is greater than GF - - - 56

133. *Of ſtrait lines drawn from the ſame point to the circumference of a circle, the greateſt is that paſſing through its center, and the leaſt falls in the oppoſite point to the greateſt*



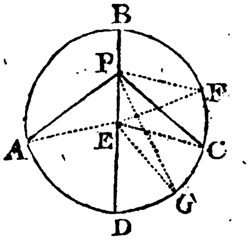


From any point P, beſide the center, let the ſtrait lines PE, PD be drawn to the circumference of a circle ABDE, meeting it in A and B, whereof PE paſſes thro' the center C; then PE is greater then PD, and PA is leſs than PB.

For join BC, CD.

1. CA, CB, CD, CE are equal - - - 30
Th. PC + CD is equal to PE - - - 49
But PC + CD is greater than PD - - - 74
Th. PE - - - is greater than PD - - - 56
2. PA + AC is leſs than PB + BC, fig. 1. - 74
Th. PA + AC is leſs than PB + AC - - 56
Th. PA - - - is leſs than PB - - - 50
3. BC - - - is leſs than PB + PC, fig. 2. - 74
Th. PA + PC is leſs than PB + PC - - 56
Th. PA - - - is leſs than PB - - - 50

134. *Only two equal lines can be drawn to the circumference of a circle, from any point which is not its center.*



Let P be a point beſide the center of the circle ABCD, then only two equal ſtrait lines can join the point P and the circumference.

For find E the center of the circle, join PE, EA, make the angle PEC equal to the angle PEA, and draw AP, PC.

Now PE, EA are equal to PE, EC each to each. - -

And the contained angles PEA, PEC are equal - con.

Th. PA and PC are equal - - - - 62.

Again, take the points F, G, on each ſide of C in the ſemicircle BCD, and join PF, PG, GE, EF.

Then PE, EG are equal to PE, EC each to each.

But the contained angle PEG is greater than PEC. 52.

Th. PG is greater than PC - - - - 75

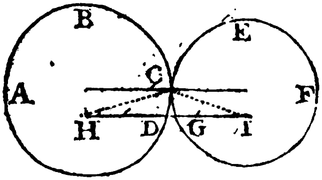
By the ſame reaſon PC is greater than PF - - -

Th. only PC is equal to PA - - - - 48

135. Corollary.

One circle can cut another, in two points only.

136. *If two circles touch one another, a ſtrait line joining their centers will fall in the point of contact.*

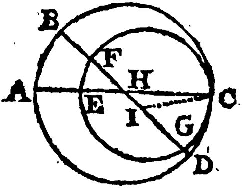


If the circles ABCD, CEFG touch outwardly in the point C; then a ſtrait line joining their centers, will fall in the point of contact C.

For if not, let H the center of the circle ABCD, and I the center of the circle CEFG, be joined by the ſtrait line HI cutting the circles in D and G, and join HC, CI.

Now HD is equal to HC - - - - - 30
And GI is equal to CI - - - - - 30
Th. HD + GI is equal to HC + CI - - 49
But HI is greater than HD + GI - - 52
Th. HI is greater than HC + CI - - 56
Which is impoſſible - - - - - 74
Therefore, a ſtrait line joining the centers muſt fall in the point of contact.

136. Caſe 2. *When the circles touch inwardly.*



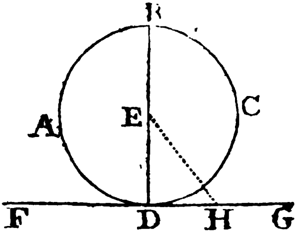
Let the circles ABCD, EFCG touch inwardly in the point C; then a ſtrait line joining their centers, will fall in the point of contact C. For if not, let H the center of ABCD, and I the center of EFCG be joined by the ſtrait line HD, cutting the circles in G and D, and draw IC.

Now HI + IC is greater than HC - - 74
And HD is equal to HC - - - - 30
Th. HI + IC is greater than HD - - - 56
Or HI + IC is greater than HI + ID - - -
Th. - - - IC is greater than - - - ID - - 50
But IG is equal to IC - - - - 30
Th. IG is greater than ID - - - - 56
Which is abſurd - - - - - - 52
Therefore, a ſtrait line joining the centers, muſt fall in the point of contact.

137. Corollary.

One circle can touch another, in one point only.

138. *A ſtrait line drawn at right angles to the extremity of a diameter, is a tangent to the circle.*



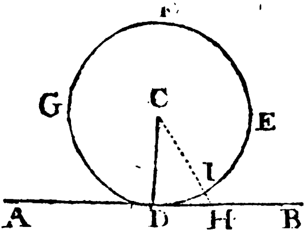
Let ABCD be a circle, whoſe center is E and diameter BD, and thro' the point D, let the ſtrait line FG be drawn at right angles to BD; then the line FDG is a tangent to the circle.

For in FG aſſume the point H, and join EH.
Now EDH is a right angle - - - theo.
Th. the angle EDH is equal to EHD + DEH - 88
Th. the angle EDH is greater than EHD - - 52
Th. the line EH is greater than the line ED - 73
Th. the point H is without the circle - - 30
Th. the line FDG is a tangent to the circle. - 38

139. Corollary.

A ſtrait line can touch a circle in one point only.

140. *A line drawn from the center of a circle to the point of contact, will be perpendicular to the tangent.*

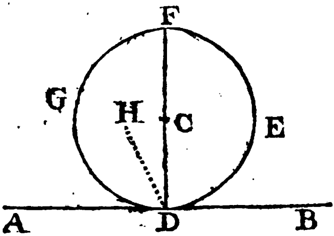


Let the ſtrait line AB touch the circle DEFG in the point D, and from the center C draw CD; then the line CD is perpendicular to the tangent AB.

For if not, upon AB let fall the perpendicular CH by problem 7, cutting the circumference in I.

Now CHD is a right angle - - - - 9
Th. the angle CHD is equal to CDH + HCD - - 88
Th. the angle CHD is greater than the angle CDH. 52
Th. the line CD is greater than the line CH. 73
But the lines CD, CI are equal - - - - 30
Th. CI is greater than CH - - - 56
Which is abſurd - - - - - 52
Th. CD muſt be perpendicular to AB - - -

141. *A line drawn at right angles to a tangent, at the point of contact, will paſs thro' the center of the circle.*

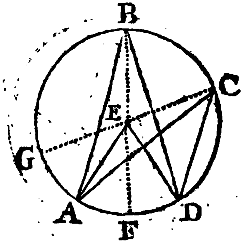


Let the tangent AB touch the circle DEFG in D, and let DF be drawn at right angles to AB: then the line DF paſſes thro' the center of the circle C.

For if not, let H be the center, and join HD.

Now HD is perpendicular to AB - - - - 140
Th. ADH is a right angle - - - - 9
Alſo ADF is a right angle - - - theo.
Th. the angles ADH, ADF are equal - - 54
Which is impoſſible - - - - - - 52
Therefore, the perpendicular DF muſt paſs thro' e the center of the circle.

142. *An angle at the center of a circle is double to an angle at the circumference, ſtanding upon the ſame arch.*

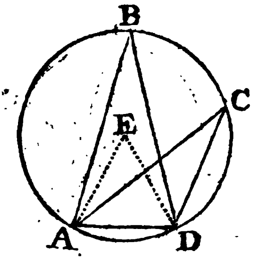


In the circle ABCD, let the angle AED at the center, and the angle ABD at the circumference, ſtand upon the arch AD; then the angle AED is double to the angle ABD.

For draw BE, and produce it to F.

Now EA is equal to EB - - - - 30
Th. the angle ABE is equal to EAB - - 63
Th. the angle ABE + EAB is double to ABE - 49
But the angle AEF is equal to ABE + EAB - 86
Th. the angle AEF is double to ABE - - 56
Alſo, the angle FED is double to EBD, by the ſame reaſon - - - - - - - -
Th. the angle AED is double to ABD - - 49
Again, let the angle ACD be taken lower, and produce the line CE to G.
Then the angle GED is double to GCD - - above
And the angle GEA is double to GCA - - above
Th. the angle AED is double to ACD - - 50

143. *All angles in the ſame ſegment of a circle, are equal to one another.*

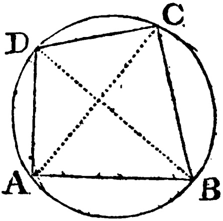


Let the circle ABCD be cut by the right line AD, and join AB, BD, AC, CD: then the angles ABD, ACD being in the ſame ſegment, are equal.

For find E the center of the circle by problem 32, and draw AE, ED.

Now the angle AED is double to ABD - - 142
And the angle AED is double to ACD - - 142
Th. each of the angles ABD, ACD is equal to half the angle AED - - - - - - -
Th. the angles ABD, ACD are equal - - 48

144. *The oppoſite angles of any quadrangle inſcribed in a circle, are equal to two right angles.*



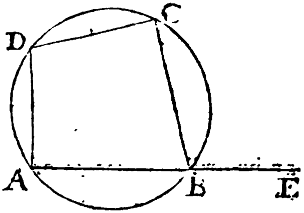
Let a circle have a quadrangle ABCD inſcribed in it: then its oppoſite angles both together, are equal to two right angles.

For join AC, BD.

Now the angle ABD - - - - = ACD - - 143
And the angle DBC - - - - = DAC - - 143
Th. the angle ABC - - - = ACD + DAC - 49
Th. the angle ABC + CDA = ACD + DAC + CDA. 49
But two right angles - - - = ACD + DAC + CDA. 86
Th. the angle ABC + CDA = two right angles - 48
And by the ſame reaſon, the angle BAD + DCB is equal to two right angles.

145. Corollary.

If any ſide of a quadrangle inſcribed in a circle is produced, the outward angle will be equal to the inward and oppoſite angle.



For let the ſide AB of a quadrangle ABCD inſcribed in a circle, be produced to E.

Now the angle ABC + CBE = two right angles - 67

And the angle ABC + ADC = two right angles 144

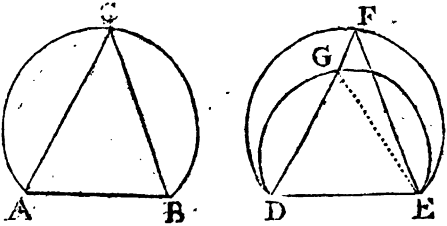
Th. the angle ABC + CBE = ABC + ADC - 48

Th. the angle - - - - CBE = ADC 56

146. Corollary 2.

If two oppoſite angles of a quadrangle taken together be equal to two right angles, a circle may be circumſcribed about the figure.

147. *Similar ſegments of circles upon equal baſes, are equal to one another.*



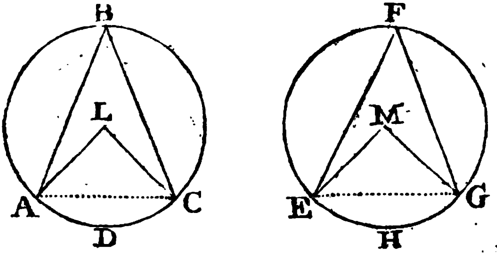
Let ABC, DEF be ſimilar ſegments of circles, conſtituted upon equal baſes AB, DE: then the ſegments ABC, DEF are equal.

For conceive the ſegment ABC applied to DEF, with the point A on the point D, and the line AB on the line DE, then the point B will agree with E, becauſe AB is equal to DE - - - - - - theo.

Now if it be poſſible, let the ſegment ABC diſagree with the ſegment DEF, and fall into a different poſition DGE, and join AC, CB, DF, FE, EG.

Now the angle DGE is equal to ACB - - 143
And the angle DFE is equal to ACB - - 36
Th. the angle DGE is equal to DFE - - 48
Which is impoſſible - - - - - 71
Th. the ſegment ABC muſt agree with DEF - -
Th. the ſegments ABC, DEF are equal - - 53

148. *In equal circles, equal angles either at the centers or circumferences, intercept equal arches.*



In the equal circles ABCD, EFGH, let the angles ABC, EFG at the circumferences be equal: then the intercepted arches ADC, EHG, are alſo equal.

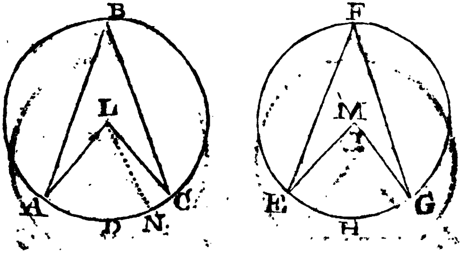
For find L and M the centers of the circles, and join AL, LC, CA, EM, MG, GE.

Now the angle ALC is double to ABC - - 142
And the angle EMG is double to EFG - - 142
Th. the angle ALC is equal to EMG - - 51
And the ſides AL, LC are equal to EM, MG - - 55
Th. the baſes AC, EG are equal - - - 62
Th. the ſegments ABC, EFG are equal - - 147
Th. the arches ADC, EHG are equal - - 50

149. Corollary.

In equal circles, the ſectors of equal angles are equal to one another, namely, ALCDA = EMGHE - - 53

150. *In equal circles, angles ſtanding upon equal arches are equal, whether at the centers or circumference.*



In the equal circles ABCD, EFGH, let the arches ADC, EHG be equal: then the angles ABC, EFG conſtituted thereon are alſo equal.

For find L and M the centers of the circles, join AL, LC, EM, MG, and if the angles ALC, EMG are not equal, make the angle ALN equal to EMG. - p. 9

Now the circle ABCD is equal to EFGH - - theo.

And the angle ALN is equal to EMG - - con.

Th. the arch ADN is equal to EHG - - 148

But the arch ADC is equal to EHG - - theo.

Th. the arch ADN is equal to ADC - - 48

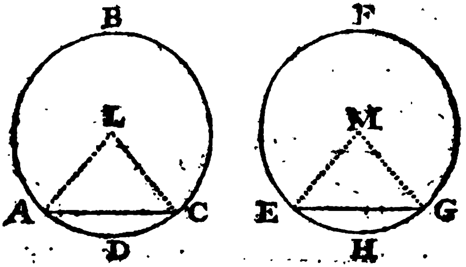
Which is impoſſible - - - - - 52

Th. the angles ALC, EMG muſt be equal - - 1

And the angles ABC, EFG are the halves of ALC, EMG - - - - - - - 142

Th. the angles ABC, EFG are equal - - 51

151. *In equal circles, equal ſtrait lines cut off equal parts of the circumferences, the leſſer arch equal to the leſſer, and the greater arch equal to the greater.*

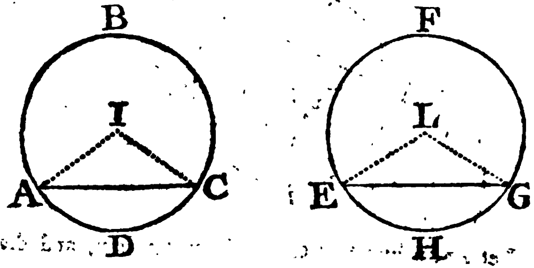


Let ABCD, EFGH be equal circles, and AC, EG be equal ſtrait lines cutting off the leſſer parts ADC, EHG of their circumferences: then the arch ADC is equal to the arch EHG, and the arch ABC is equal to EFG.

For find L and M the centers of the circles, and join AL, LC, EM, MG.

Now the ſides AL, LC are equal to EM, MG - - 55
And the baſes AC, EG are equal - - theo.
Th. AL, LC, CA = EM, MG, GE each to each, - -
Th. the angles ALC, EMG are equal - - 56
Th. the arch ADC is equal to the arch EHG - 148
And the arch ABC is equal to the arch EFG - 50

152. *In equal circles, equal arches are ſubtended by equal ſtrait lines.*

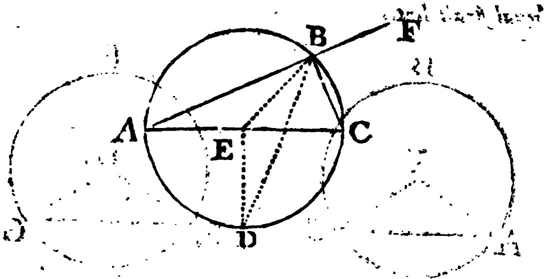


Let the circles ABCD, EFGH be equal, and the parts ADC, EHG of their circumferences be equal, and join AC, EG; then the ſtrait lines, or chords AC, EG are likewiſe equal.

For find I and L the centers of the circles, and join AI, IC, EL, LG.

Now the circles ABCD, EFGH are equal - - theo.
And the arches ADC, EHG are equal - - theo.
Th. the triangles AIC, ELG have the ſides AI, IC equal to EL, LG - - - - - - 55
And the contained angles AIC, ELG are equal - 150
Therefore, the chords AC and EG are equal - 62

153. *An angle in a ſemicircle is a right angle.*



Let ABCD be a circle, whoſe center is E, and diameter is AC, and to any point B in the circumference draw AB, CB: then ABC is a right angle.

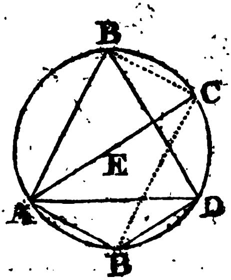
For join EB, and produce AB to F.

Now EA, EB, EC are equal - - - - 50
Th. the angle ABE is equal to EAB - - 63
And the angle EBC is equal to BCE - - 63
Th. the angle ABC is equal to EAB + BCE - 49
But the angle CBF is equal to EAB + BCE - 86
Th. the angle ABC is equal to CBF - - 48
Th. ABC is a right angle - - - - 9

Otherwiſe, draw ED at right angles to AC, and join BD.

Here, the angle ABD is equal to half AED - - 142
And the angle DBC is equal to half DEC - - 142
But AED, DEC are right angles - - con.
Th. ABC is a right angle - - - - 52

154. The angle in the segment of a circle differs from a right one, by the angle contained between the base of the segment and a diameter drawn from its extremity.



Let the circle ABCD be cut by the right line AB, and draw AB, BD, and the diameter AC: then the angle ABD differs from a right angle, by DAC.

For join BC.

Now the angle ABC, is in a semicircle - - - - 33

Th. ABC is a right angle - - - - - - 153

And ABD differs from it by the angle DBC - - ā

But the angle DAC is equal to the angle DBC - - 143

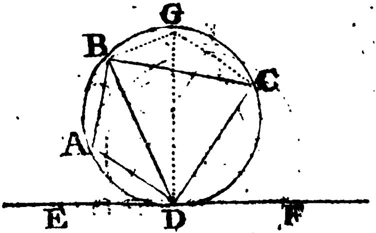
Therefore, the angle ABD differs from a right one, by the angle DAC - - - - - - - - - 56

Remark.

An angle in the greater segment of a circle is acute, but in the lesser it is obtuse.

155. *The*

[illegible] *The angles made by a tangent and a line drawn from the point of contact cutting the circle, are equal to the angles contained in the opposite segments of the circle.*



Let the circle ABCD have a tangent EF touching it in D, draw DB at pleasure cutting the circle, and join BC, CD, DA, AB; then the angles EDB, BDF are equal to the angles BCD, DAB each to each.

For draw DG at right angles to EF, and join BG, GC.
Now the line DG is a diameter of the circle - 141
Th. GCD is a right angle - - - 153
Th. the angle EDG - - - = GCD - - 54
But the angle BDG - - - - = GCB - - 143
Th. the angle EDB - - - = BCD - - 50
Ag. the angle EDB + BDF = 2 right angles - 67
And the angle BCD + DAB = 2 right angles - 144
Th. the angle EDB + BDF = BCD + DAB - - 48
Th. the angle - - - - BDF = - - - DAB - - 50

BOOK

BOOK IV.

156. *Quantity is that which is compoſed of parts.*

157. *A multiple of a quantity, is the ſaid quantity taken any number of times.*

158. *Ratio, is a mutual relation between two quantities of one kind, according to what part or parts, the one compriſes of the other.*

159. *Ratios are equal, when the quantities are the ſame part or parts of one another, in each pair.*

Thus, the ratio between 3 *and* 6, *is equal to the ratio between* 4 *and* 8; *and the ratio between* 1 *and* 3, *is equal to the ratio between a foot and a yard.*

160. *Proportion is an equality of ratios.*

Thus the numbers 3, 6, 4, 8 *are proportional; and if* A, B *repreſent quantities of one kind, and* C, D *other quantities of one kind, having an equal ratio with* A *and* B, *thoſe four quantities are proportional, and wrote* A : B :: C : D, *and read, as* A *to* B, *ſo is* C *to* D.

Note. *The firſt and third terms or quantities in a proportion, are called antecedents, and the ſecond and fourth conſequents.*

161. *A mean proportional, is a quantity that takes the ſecond and third places in a proportion. Thus, if* A *is to* B, *as* B *to* C, *the quantity* B *is a mean proportional between* A *and* C.

162. *Alternate ratio, is when the antecedents are compared together, and the conſequents together.*

163. *Compounded ratio, is when the ſum of the antecedent and conſequent, is compared with either of them.*

164. *Divided ratio, is when the difference of the antecedent and conſequent, is compared with either of them.*

165. *Similar figures are thoſe which be equiangular, and have the ſides about the equal angles proportional.*

Axioms.

166. *Equal quantities have equal ratios to the ſame quantity, or to equal quantities.*

167. *Quantities that have equal ratios to the ſame quantity, or to equal quantities, are equal to one another.*

168. *If two quantities be compared with any third quantity, the greater of them will have a greater ratio to it, than the leſſer.*

169. *If two quantities be compared with any third quantity, that which has the greater ratio to it, is the greater quantity.*

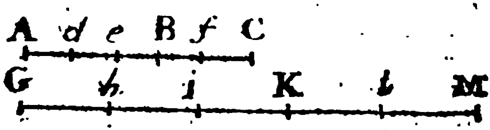
170. *Any quantity may be divided into equal parts, that ſhall be leſs than any given quantity of the ſame kind.*

171. *Equimultiples of equal quantities are equal.*

172. *Any multiple of the whole, is equal to the ſame multiple of all its parts taken together.*

173. *Ratios that are equal to the ſame ratio are equal to one another, and may be taken the one for the other.*

174. *Quantities are proportional to their equimultiples.*



Let AB, BC be quantities of the ſame kind, and n be any number whatſoever; then AB : BC :: AB × n : BC × n.

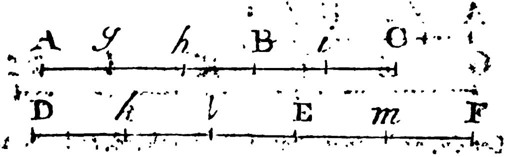
For conceive AB, BC divided into parts equal to one another at d, e, f, &c. and alſo that G h, h i, i K, &c. are equal to n times A d, d e, e B, &c.

Now A d, d e, e B, &c. are equal - - con.
Th. G h, h i, i K, &c. are equal - - 171
And the number of parts in GK, KM are equal to the number in AB, BC each to each - - con.
Th. AB, BC and GK, KM have equal ratios - 159
Th. AB : BC :: GK : KM - - - - 160
But AB × n = GK, and BC × n = KM - - 172
Th. AB : BC :: AB × n : BC × n - - 56

Remark.

If A *and* B *repreſent quantities of one kind, then* A : B :: n A : n B.

175. *If four quantities of the ſame kind are proportional, they will be alternately proportional*



Let AB, BC, DE, EF be quantities of the ſame kind, and AB : BC :: DE : EF; then AB : DE :: BC : EF.

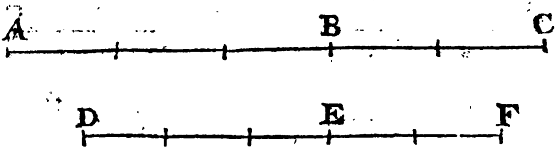
For conceive AB, BC divided into parts equal to one another at g, h, i, &c. and alſo DE, EF divided into the ſame number of equal parts with AB, BC reſpectively, at k, l, m, &c.

Now AB : BC :: DE : EF - - - - - theo.
Th. AB, BC and DE, EF have equal ratios - 166
Th. the parts in DE, EF are of equal magnitude 159
Th. A g : D k :: AB : DE - - - - 174
And A g : D k :: BC : EF - - - - 174
Th. AB : DE :: BC : EF - - - - 173

Remark.

If A, B, C, D *repreſent quantities of one kind, and* A : B :: C : D, *then* A : C :: B : D.

176. *If four quantities are proportional, they will alſo be proportional by compoſition, or addition.*



If the quantities AB, BC, DE, EF be proportional, and AB : BC :: DE : EF; then AB : AC :: DE : DF.

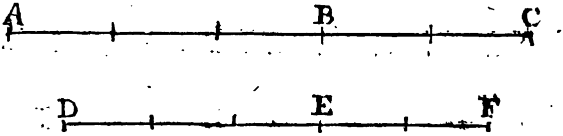
For conceive AB, BC divided into parts equal to one another, and DE, EF divided into the ſame number of equal parts with AB, BC reſpectively.

Now AB : BC :: DE : EF - - - - then
Th. AB, BC and DE, EF have equal ratios - - 160
Th. the parts in DE, EF are of equal magnitude 159
Th. AB, AC and DE, DF have equal ratios - - 159
Th. AB : AC :: DE : DF - - - - - 160

Remark.

If A, B, C, D *repreſent proportional quantities, and* A : B :: C : D, then A : A + B :: C : C + D.

177. *If four quantities are proportional, they will alſo be proportional by diviſion, or ſubſtraction.*



Let the quantities AC, AB, DF, DE be proportional, and AC : AB :: DF : DE; then AB : BC :: DE : EF.

For conceive AB, BC divided into parts equal to one another, and DE, EF divided into the ſame number of equal parts with AB, BC, each to each.

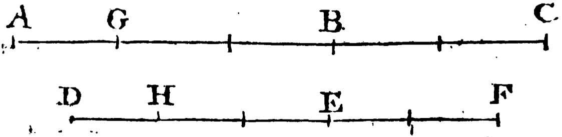
Now AC : AB :: DF : DE - - - - - theo.
Th. AC, AB and DF, DE have equal ratios - - 160
Th. the parts in DF, DE are of equal magnitude 159
Th. AB, BC and DE, EF have equal ratios - - 159
Th. AB : BC :: DE : EF - - - - - 160

Remark.

If A, B, C, D *repreſent proportional quantities,* A *greater than* B, *and* A : B :: C : D, *then* A — B : B :: C — D : D.

178. *If*

178. *If four quantities are proportional, the ſum and difference of the firſt and ſecond will be proportional to the ſum and difference of the third and fourth.*



Let the quantities AB, BC, DE, EF be proportional AB : BC :: DE : EF, and let AB be greater than BC; take BG, EH equal to BC, EF reſpectively, then AC, AG and DF, DH are the ſums and differences; I ſay, AC : AG :: DF : DH.

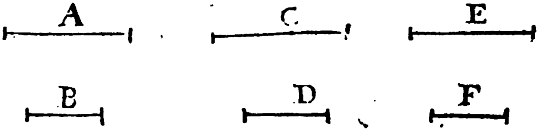
For conceive AG, GB, BC divided into parts equal to one another, and DH, HE, EF divided into the ſame number of equal parts with AG, GB, BC each to each.

Now AB : BC :: DE : EF - - - - theo.
Th. AB, BC and DE, EF have equal ratios - - 160
Th. the parts in DH, HE, EF muſt be of equal magnitude - - - - - - - - - 159
Th. AC, AG and DF, DH have equal ratios - - 159
Th. AC : AG :: DF : DH - - - - 160

Remark.

If A, B, C, D *repreſent proportional quantities,* A *greater than* B, *and* A : B :: C : D, *then* A + B : A — B :: C + D : C — D.

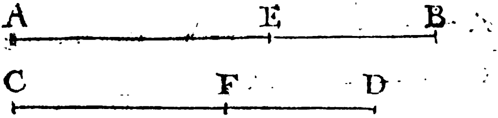
179. *If any number of quantities of one kind are proportional, as one antecedent to its conſequent, ſo is the ſum of all the antecedents to the ſum of all the conſequents.*



Let A, B, C, D, E, F be quantities of one kind, and A : B :: C : D :: E : F; then A : B :: A + C + E : B + D + F.

For A : C :: B : D - - - - - 175
Th. A : A + C :: B : B + D - - - - 176
Th. A : B :: A + C : B + D* - - - 175
But A : B :: E : F - - - - - theo.
Th. A + C : B + D :: E : F - - - - 173
Th. A + C : E :: B + D : F - - - 175
Th. A + C : A + C + E :: B + D : B + D + F. 176
Th. A + C : B + D :: A + C + E : B + D + F.* 175
Ag. A + C : B + D :: A : B - - - above
Th. A : B :: A + C + E : B + D + F - - 173

180. *If one quantity be to another, as a part of the one to a part of the other; thoſe quantities and their remaining parts are proportional, and alſo the parts themſelves.*



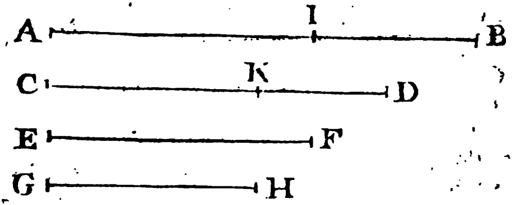
Let the quantities AB, CD be ſo divided at E and F, that AB : CD :: AE : CF; then AB : CD :: EB : FD, and AE : CF :: EB : FD.

Now AB : CD :: AE : CF* - - - - theo.
Th. AB : AE :: CD : CF - - - - 179.
Th. AB : EB :: CD : FD - - - - 177
Th. AB : CD :: EB : FD* - - - - 175.
And AE : CF :: EB : FD - - - - 179.

Remark.

If A, B, C, D *repreſent ſuch quantities that* A + B : C + D :: B : D; *then* A + B : C + D :: A : C, *and* A : B :: C : D.

181. *If four quantities of one kind be proportional, the greatest and least taken together, are greater than the other two taken together.*

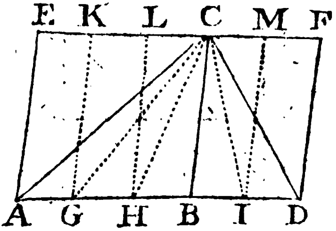


Let AB, CD, EF, GH be quantities of one kind, AB the greatest, GH the least, and AB : CD :: EF : GH; then AB + GH is greater than CD + EF.

For in AB, CD take AI, CK equal to EF, GH each to each.

Now AB : CD :: EF : GH - - - - - theo.
Th. AB : CD :: AI : CK - - - - - 56
Th. AB : CD :: IB : KD - - - - - 180
Th. AB, CD and IB, KD have equal ratios - - 160
But AB is greater than CD - - - - - theo.
Th. IB is greater than KD - - - - 159
Ag. AI + GH is equal to EF + CK - - 49
Th. AI + GH + IB is greater than EF + CK + KD. 49
Or, AB + GH is greater than CD + EF - - -

182. *Triangles and parallelograms having the ſame altitude, are proportional to their baſes.*



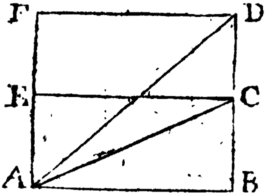
Let the triangles ACB, BCD and the parallelograms EB, BF be conſtituted between the parallels AD, EF; then AB : BD :: ACB : BCD, and AB : BD :: EB : BF.

For conceive the baſes AB, BD divided into parts equal to one another at G, H, I, &c. let GK, HL, IM be drawn parallel to AE or BC, and join GC, CH, CI.

Now the parts AG, GH, HB, &c. are equal - con.
Th. the triangles ACG, GCH, &c. are equal - 96
And the parallegr. AK, GL, &c. are equal - 94
Th. AB, BD and ACB, BCD have equal ratios - 159
Alſo, AB, BD and EB, BF have equal ratios - 159
Th. AB : BD :: ACB : BCD - - - - 160
And AB : BD :: EB : BF - - - - 160

183. Corollary.

Triangles and parallelograms having the ſame baſe, are proportional to their altitudes.



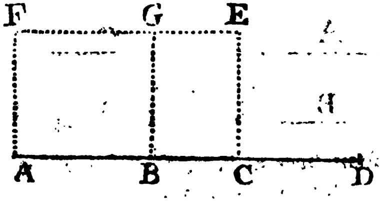
Since all parallelograms upon the ſame baſe, and between the ſame parallels, are equal to each other; and all triangles upon the ſame baſe, and between the ſame parallels, are likewiſe equal to one another, by articles 93 and 95; let therefore the triangles ABC, ABD, and parallelograms BE, BF be right angled at B; then AB is their common baſe, and BC, BD are their reſpective altitudes. - - - - - 29

Now if BC, BD be conſidered as baſes, the triangles and parallelograms have the ſame altitude AB. - 29

Th. BC : BD :: ABC : ABD - - - 182

And BC : BD :: BE : BF - - - 182

184. Of any three ſtrait lines, the firſt is to the ſecond, as a rectangle contained by the firſt and third, to a rectangle contained by the ſecond and third.

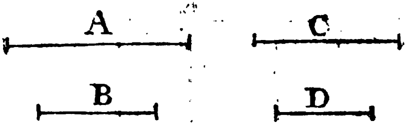


Let the ſtrait lines AB, BC, CD be taken at pleaſure; I ſay, AB : BC :: AB × CD : BC × CD.

For let ABC be a ſtrait line, erect CE perpendicular to AC, make CE equal to CD, draw EF parallel to CA, and AF, BG parallel to CE.

Now AG, BE are rectangles - - - 92
Th. AB : BC :: AG : BE - - - 182
Or AB : BC :: AB × BG : BC × CE - - 61
But BG and CE are equal - - - - 91
Th. AB : BC :: AB × CE : BC × CE - - 56
Again, CE is equal to CD - - - con.
Th. AB : BC :: AB × CD : BC × CE - - 56

185. *If four ſtrait lines are proportional, the rectangle contained by the extremes, will be equal to the rectangle contained by the middle terms.*



Let the ſtrait-lines A, B, C, D be proportional, and $A : B :: C : D$; then $A \times D = B \times C$.

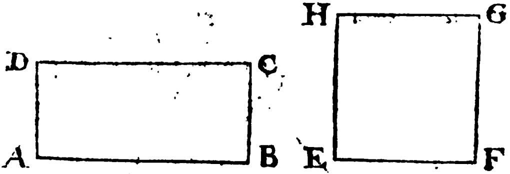
For $A : B :: A \times D : B \times D$ - - - 184
Th. $C : D :: A \times D : B \times D$ - - - 173
Ag. $C : D :: C \times B : D \times B$ - - - 184
Th. $A \times D : B \times D :: C \times B : D \times B$ - - 173
Or $A \times D : B \times D :: B \times C : B \times D$ - - 56
Th. $A \times D = B \times C$ - - - - - 167

186. Corollary.

If three ſtrait lines are proportional, the rectangle contained by the extremes, will be equal to the ſquare of the middle term.

Thus if $A : B :: B : C$ - - - - - then
Then $A \times C = B^2$ - - - - - 185

187. *The ſides of equal rectangles, are reciprocally proportional.*



Let the rectangles ABCD, EFGH be equal to each other, then AB : EF :: FG : BC.

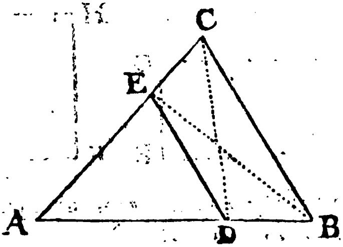
For AB : EF :: AB × BC : EF × BC - - - 184
And AB × BC = EF × FG - - - - theo.
Th. AB : EF :: EF × FG : EF × BC - - - 56
Ag. FG : BC :: EF × FG : EF × BC - - - 184
Th. AB : EF :: FG : BC - - - 173

188. Corollary.

If four ſtrait lines are proportional, they will be alternately proportional.

Thus, if AB : EF :: FG : BC - - - - theo.
Then AB × BC = EF × FG - - - 185
Th. AB : FG :: EF : BC - - - 187

189. *A line drawn within a triangle parallel to one of its ſides, will divide the other ſides proportionally.*

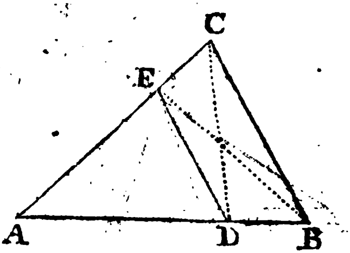


In the triangle ABC, let the line DE be drawn parallel to the ſide BC; then AD : DB :: AE : EC, and AD : AE :: DB : EC.

For join BE, CD.

Now AD : DB :: ADE : EDB* - - - 181
And AE : EC :: ADE : EDC - - - 182
But the triangles EDB : EDC are equal - - 95
Th. AE : EC :: ADE : EDB* - - 56
Th. AD : DB :: AE : EC - - 173
And AD : AE :: DB : EC - - 175

190. *If two ſides of a triangle are divided proportionally, a line joining the points of diviſion will be parallel to the other ſide.*



Let ABC be a triangle, whoſe ſides AB, AC are divided proportionally at D and E, ſo that AD : DB :: AE : EC, and join DE; then DE is parallel to BC.

For join BE, CD.

Now AD : DB :: AE : EC - - - - - there

And AD : DB :: ADE : EDB - - - - - 182

Th. AE : EC :: ADE : EDB - - - - - 173

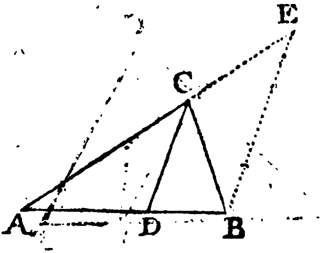
Ag. AE : EC :: ADE : EDC - - - - - 184

Th. ADE : EDB :: ADE : EDC - - - - - 173

Th. the triangles EDB, EDC are equal - - - 167

Th. DE is parallel to BC - - - - - 97

191. *If the vertical angle of a triangle is biſected by a line meeting the baſe, the parts of the baſe will be proportional to the ſides of the triangle.*



Let ABC be a triangle, having its vertical angle ACB divided equally by the right line CD; then AC : CB :: AD : DB.

For produce AC to E, and draw EB parallel to DC.

Now the angle ACD = CEB - - - - 85

And the angle ACD = DCB - - - - theo.

Th. the angle CEB = DCB - - - - 48

Ag. the angle CBE = DCB - - - - 82

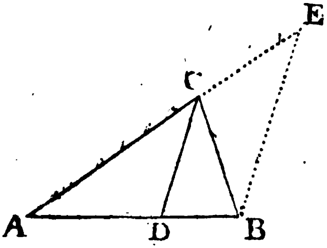
Th. the angles CEB, CBE are equal - - - 48

Th. the ſides BC, CE are equal - - - 65

But AC : CE :: AD : DB - - - - 189

Th. AC : CB :: AD : DB - - - - 56

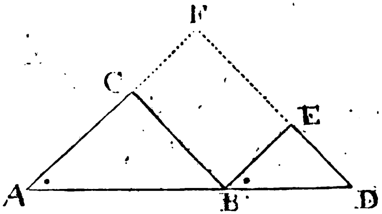
192. *If the baſe of a triangle is divided in proportion to the ſides, a line joining the vertex and point of diviſion will biſect the vertical angle.*



Let the triangle ABC have its baſe ſo divided in D, that AC : CB :: AD : DB, and join CD; then the angles ACD, DCB are equal.

For produce AC to E, and draw BE parallel to DC - - - - - - - - p. 10
Now AC : CE :: AD : DB - - - - - 189
And AC : CB :: AD : DB - - - - - theo.
Th. AC : CE :: AC : CB - - - - - 173
Th. CE is equal to CB - - - - - 167
Th. the angle CBE = CEB - - - - - 63
But the angle ACD = CEB - - - - - 83
Th. the angle CBE = ACD - - - - - 48
Ag. the angle CBE = DCB - - - - - 82
Th. the angle ACD = DCB - - - - - 48

193. *In equiangular triangles, the ſides about the equal angles are proportional, if taken correſpondently.*

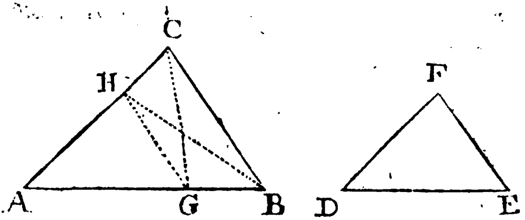


Let ABC, BDE be equiangular triangles, having the angle ABC equal to the angle BDE, and the angle BAC equal to DBE; then AB : AC :: BD : BE, AB : BC :: BD : DE, and AC : CB :: BE : ED.

For let ABD be a ſtrait line, and produce AC, DE, to F.

Now AF is parallel to BE, and DF is parallel to BC. 80
Th. BCFE is a parallelogram - - - - 22
Th. BC is equal to EF, and BE is equal to CF - 91
But AB : AC :: BD : CF - - - - 189
And AB : FE :: BD : ED - - - - 189
Th. AB : AC :: BD : BE - - - - 56
And AB : BC :: BD : DE - - - - 56
Alſo AC : CB :: BE : ED by the ſame reaſon -

193.* Otherwiſe.



Let the triangles ABC, DEF be equiangular, having the angles ABC, CAB equal to the angles DEF, FDE each to each: then AB : DE :: AC : DF, AB : DE :: BC : EF, and AC : DF :: CB : FE.

For let AB be greater than DE, make AG, AH equal to DE, DF each to each, and join BH, HG, GC.

Now the angle AGH is equal to DEF - - 62

Th. the angle AGH is equal to ABC - - 48

Th. GH is parallel to BC - - - - - 80

Th. the triangles GHB, GHC are equal - - 95

Th. the triangles ABH, AGC are equal - - 49

But AB : AG :: ABH : AGH - - - 182

Th. AB : AG :: AGC : AGH - - - 56

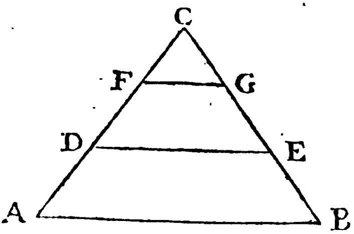
Ag. AC : AH :: AGC : AGH - - - 182

Th. AB : AG :: AC : AH - - - 173

Th. AB : DE :: AC : DF - - - - 56

Alſo AB : DE :: BC : EF } by the ſame reaſon.
And AC : DF :: CB : FE }

194. *If lines be drawn parallel to the baſe of a triangle cutting the ſides, the correſponding ſegments will be proportional.*



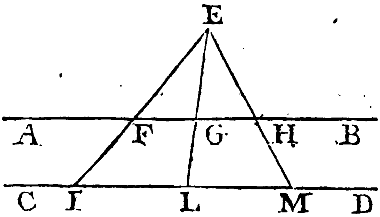
In the triangle ABC, let the lines DE, FG be drawn parallel to the baſe AB; then FD : DA :: GE : EB.

For the angles CAB, CDE, CFG are equal - - 83
And the angles CBA, CED, CGF are equal - - 83
Th. all the triangles are equiangular - - 27
Th. CF : CG :: CD : CE - - - - 193
And CF : CG :: FD : GE - - - - 189
Th. CD : CE :: FD : GE - - - - 173
Alſo CD : CE :: DA : EB - - - - 189
Th. FD : GE :: DA : EB - - - - 173
Th. FD : DA :: GE : EB - - - - 175

Remark.

Hence again, if four right lines are proportional, they will alſo be proportional either compounded or divided.

195. *If parallels be cut by many ſtrait lines proceeding from a point without them, their correſponding ſegments will be proportional.*



Let the parallels AB, CD be cut by ſtrait lines proceeding from the point E, in F, G, H, I, L, M; then FG : IL :: GH : LM.

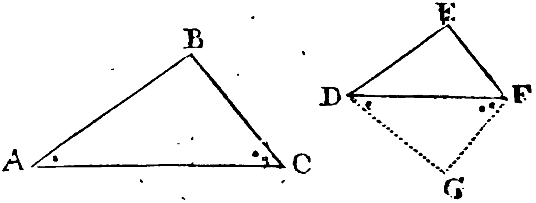
For the angles EFG, EGF are equal to EIL, ELI. 83
And the angle IEL is common - - - -
Th. the triangles EFG, EIL are equiangular - - 27
By the ſame reaſon, the triangles EGH, ELM are equiangular - - - - - - - - -
Th. EG : EL :: FG : IL - - - - 193*
And EG : EL :: GH : LM - - - - 193*
Th. FG : IL :: GH : LM - - - - 173

196. Corollary.

If one of the parallels is divided into equal parts, every other parallel will likewiſe be divided into equal parts.

For if FG = GH, then IL = LM - - 167

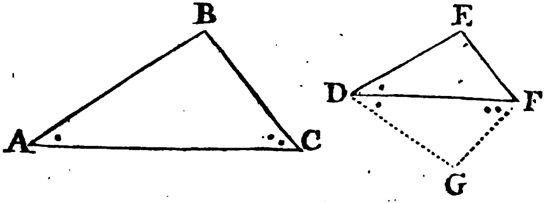
197. *Triangles whoſe ſides are all proportional, are equiangular, having their correſponding angles equal to each other.*



Let the triangles ABC, DEF have their ſides proportional, namely, BA : AC :: ED : DF, AB : BC :: DE : EF, and AC : CB :: DF : FE; then the angles BAC, ABC, ACB are equal to EDF, DEF, DFE each to each.

For make the angle FDG equal to the angle BAC, and the angle DFG equal to ACB - - p. 9
Hence, the angle DGF = ABC - - - - 87
Th. the triangles ABC, DGF are equiangular - - 27
Th. BA : AC :: GD : DF - - - - - 193
And BA : AC :: ED : DF - - - - th. q.
Th. GD : DF :: ED : DF - - - - - 173
Th. GD is equal to ED - - - - - 167
By the ſame reaſon GF is equal to FE - - -
Th. the angle FDG = EDF, DGF = DEF, and DFG = DFE - - - - - - - 66
Th. the angle BAC = EDF, ABC = DEF, and ACB = DFE - - - - - - - 48

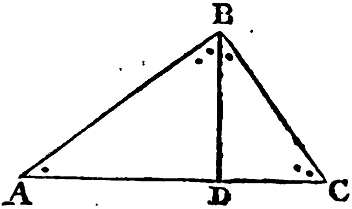
198. *If triangles have an angle of the one equal to an angle of the other, and the ſides about them proportional, thoſe triangles are equiangular, having the angles equal which are oppoſite to correſponding ſides.*



Let the triangles ABC, DEF have the angle BAC equal to the angle EDF, and BA : AC :: ED : DF; then the angles ABC, ACB are equal to DEF, DFE each to each.

For make the angle FDG equal to the angle BAC, and the angle DFG equal to ACB - - p. 9
Now the angle ABC = DGF - - - - 87
Th. the triangles ABC, DGF are equiangular - 27
Th. BA : AC :: GD : DF - - - - 193
But BA : AC :: ED : DF - - - - theo.
Th. GD : DF :: ED : DF - - - - 173
Th. GD is equal to ED - - - - 167
And the angle GDF is equal to FDE - - 48
Th. the angle DEF = DGF, and DFE = DFG - 62
Th. the angle ABC = DEF, and ACB = DFE - 48

199. *In right angled triangles, a perpendicular falling from the right angle upon the baſe, will be a mean proportional between the ſegments made thereby, and each ſide will be a mean proportional between the baſe and adjacent ſegment.*



If the triangle ABC has a right angle at B, and the line BD be drawn perpendicular to the baſe AC; then AD : DB :: BD : DC, AC : AB :: AB : AD, and AC : CB :: BC : CD.

For the angle ABC is equal to ADB - - 54

And the angle BAC is common - - - -

Th. the angle ACB is equal to ABD - - 87

By the ſame reaſon the angle BAC is equal to DBC. -

Th. the three triangles are equiangular - - 27

Th. AD : DB :: BD : DC - - - - 193

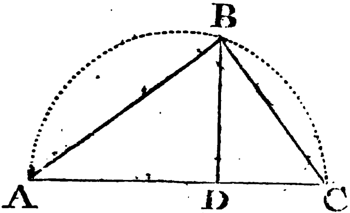
- - - AC : AB :: AB : AD - - - - 193

And AC : CB :: BC : CD - - - - 193

Remark.

Remark.

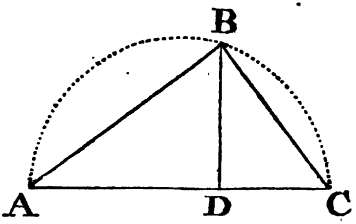
Hence again, it is eaſy to demonſtrate, that in right angled triangles; the ſquare of the ſide oppoſite to the right angle, is equal to both the ſquares of the ſides containing it, namely, $AC^2 = AB^2 + BC^2$.



For $AC \times AD = AB^2$ - - - - 186
And $AC \times CD = BC^2$ - - - - 186
Th. $AC \times AD + AC \times CD = AB^2 + BC^2$ - 49
Or $AD \times AC + DC \times AC = AB^2 + BC^2$ - 56
But $AD \times AC + DC \times AC = AC^2$ - - 107
Th. $AC^2 = AB^2 + BC^2$ - - - - - 48

200. Corollary.

In right angled triangles, the ſquares of the ſides containing the right angle, are proportional to the ſegments of the baſe made by a perpendicular; namely, $AD : DC :: AB^2 : BC^2$.

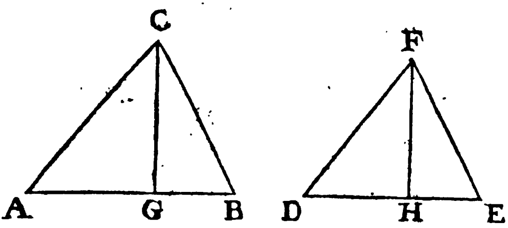


For $AB^2 = AD \times AC$ - - - - - 186
And $BC^2 = DC \times AC$ - - - - - - 186
Th. $AB^2 : BC^2 :: AD \times AC : DC \times AC$ - - 160
But $AD : DC :: AD \times AC : DC \times AC$ - - 184
Th. $AD : DC :: AB^2 : BC^2$ - - - 173

Remark.

By this theorem, it is very eaſy to find ſquares in the given ratio of AD *to* DC; *for 'tis only deſcribing a ſemicircle upon the diameter* AC, *erecting the perpendicular* DB, *and drawing* AB, BC, *which are the ſides of the ſquares required.*

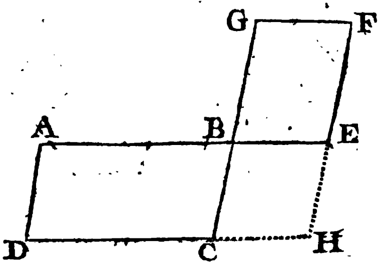
201. *In equiangular triangles, the correſponding ſides and perpendiculars are proportional.*



Let the triangles ABC, DEF be equiangular, having the angle A equal to the angle D, the angle B equal to the angle E, and CG, FH perpendiculars to the baſes AB, DE; then CG : AB :: FH : DE.

For the angle AGC is equal to DHF - - 54
And the angle CAB is equal to FDE - - theo.
Th. the angle ACG is equal to DFH - - 87
And the angle ACB is equal to DFE - - 87
Th. the correſponding triangles are equiangular 27
Th. AC : DF :: CG : FH - - - - 193
And AC : DF :: AB : DE - - - - 193
Th. CG : FH :: AB : DE - - - - 173
Th. CG : AB :: FH : DE - - - - 175

202. *Equiangular parallelograms, are proportional to the rectangles contained by their ſides.*

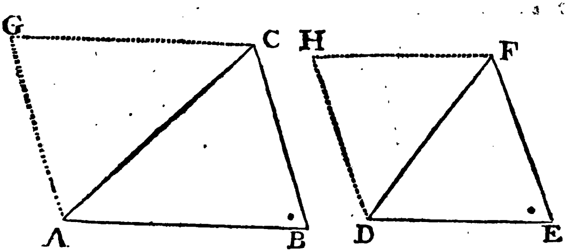


Let ABCD, GBEF be equiangular parallelograms, having the angle ABC equal to the angle GBE; then AB × BC : GB × BE :: AC : GE.

For let AB, BE be placed in a ſtrait line, and produce DC, FE to H.

Now the angle ABG + GBE = two right angles · 67
Th. the angle ABG + ABC = two right angles - 56
Th. GBC is a ſtrait line - - - - 69
Th. BEHC is a parallelogram - - - 22
Th. AB : BE :: AC : BH - - - - 182
But AB : BE :: AB × BC : BE × BC - - 184
Th. AB × BC : BE × BC : AC : BH - - 173
Th. AB × BC : AC :: BE × BC : BH - - 175
Alſo GB × BE : GE :: BE × BC : BH, by the ſame reaſon - - - - - - - -
Th. AB × BC : AC :: GB × BE : GE - - 173
Th. AB × BC : GB × BE :: AC : GE - - 175

203. *Triangles that have an angle of the one equal to an angle of the other, are proportional to rectangles contained by the ſides about thoſe equal angles.*

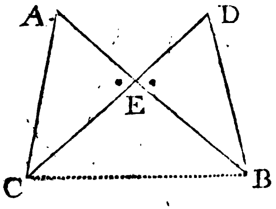


Let the triangles ABC, DEF have the angle ABC equal to the angle DEF; then AB × BC : DE × EF :: ABC : DEF.

For draw AG, CG parallel to BC, BA and DH, FH parallel to EF, ED. - - - - p. 10
Now ABCG, DEFH are parallelograms - - 22
Th. AB × BC : DE × EF :: BG : EH - - 202
But ABC, DEF are the halves of BG, EH - - 91
Th. ABC - - : DEF - - :: BG : EH - - 174
Th. AB × BC : DE × EF :: ABC : DEF - - 173.

203. Otherwiſe,

Draw the lines AB, CD *at pleaſure interſecting at* E, *and join* AC, BD; *I ſay as before,* AE × EC : BE × ED :: *triangle* AEC : *triangle* BED; *for draw* CB.



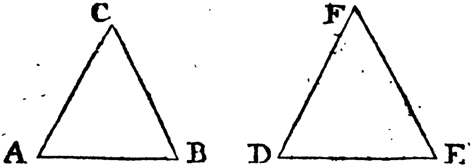
Now AE : EB :: AE × EC : EB × EC - - 184
And AE : EB :: AEC : CEB - - - 182
Th. AE × EC : EB × EC :: AEC : CEB - - 173
Th. AE × EC : AEC :: EB × EC : CEB - - 175
Alſo BE × ED : BED :: EB × EC : CEB by the ſame reaſon. - - - - - - - -
Th. AE × EC : AEC :: BE × ED : BED - 173
Th. AE × EC : BE × ED :: AEC : BED - 175

204. *Hence, if the rectangles contained by the ſides about the equal angles are equal, the triangles will be equal, and thoſe ſides reciprocally proportional.*

For let AE × EC be equal to BE × ED - - theo.
Then the triangle AEC is equal to BED - - 167
And AE : EB :: DE : EC - - - - 187

205. *Equiangular triangles, are proportional to the ſquares of their correſponding ſides.*

[illegible]
[illegible]



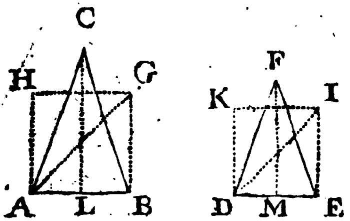
Let the triangles ABC, DEF be equiangular, having the angle ABC equal to the angle DEF, and the angle BAC equal to the angle EDF; then $AB^2 : DE^2$ :: ABC : DEF.

For AB : BC :: AB^2 : AB × BC - - 184
And DE : EF :: DE^2 : DE × EF - - 184
But AB : BC :: DE : EF - - - 193
Th. AB^2 : AB × BC :: DE^2 : DE × EF - - 173
Th. $AB^2 : DE^2$ :: AB × BC : DE × EF - - 175
Alſo ABC : DEF :: AB × BC : DE × EF - - 203
Th. $AB^2 : DE^2$ :: ABC : DEF - - - 173

205. Other-

205. Otherwiſe.

Let ABC, DEF *be equiangular triangles, having the angle* ABC *equal to the angle* DEF, *and the angle* BAC *equal to the angle* EDF; *then* $AB^2 : DE^2 :: ABC : DEF$.



For upon the baſes AB, DE let fall the perpendiculars CL, FM by problem 7, deſcribe the ſquares ABGH, DEIK by problem 11, and join AG, DI.

Now BG : CL :: ABG : ABC - - - 183

And AB is equal to BG - - - - - 24

Th. AB : CL :: ABG : ABC - - - 56

And DE : FM :: DEI : DEF, by the ſame reaſon. -

But AB : CL :: DE : FM - - - 201

Th. ABG : ABC :: DEI : DEF - - - 173

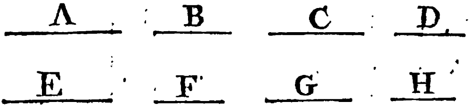
Th. ABG : DEI :: ABC : DEF - - - 175

Alſo ABG : DEI :: ABGH : DEIK - - 174

Th. ABGH : DEIK :: ABC : DEF. - - - 173

Or $AB^2 : DE^2 :: ABC : DEF$ - - - -

206. *The rectangles contained by the corresponding terms in two ranks of proportional lines, are also proportional.*



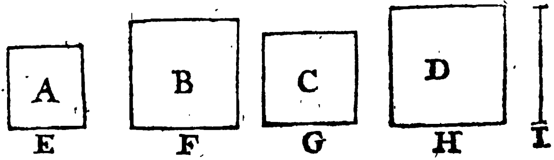
Let A, B, C, D; E, F, G, H be proportional lines, and A : B :: C : D; E : F :: G : H; then A × E : B × F :: C × G : D × H.

For A : B :: A × E : B × E - - - 184
And C : D :: C × G : D × G - - - 184
Th. A × E : B × E :: C × G : D × G - - 173
Th. A × E : C × G :: B × E : D × G * - - 175
Ag. E : F :: B × E : B × F - - - 184
And G : H :: D × G : D × H - - - 184
Th. B × E : B × F :: D × G : D × H - - 173
Th. B × E : D × G :: B × F : D × H * - - 175
Th. A × E : C × G :: B × F : D × H - - 173
Th. A × E : B × F :: C × G : D × H - - 175

207. Corollary.

The squares of proportional lines are likewise proportional.

208. *If ſquares be proportional, their ſides are alſo proportional.*

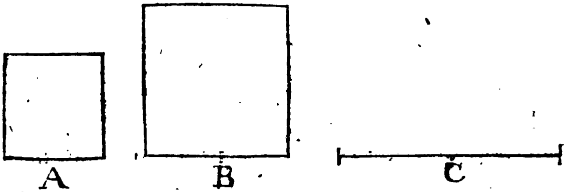


Let A, B, C, D be proportional ſquares, whoſe ſides are E, F, G, H, and A : B :: C : D; then E : F :: G : H.

For take the line I, a fourth proportional to E, F and G, by problem 5.

Now E : F :: G : I - - - - - con.
Th. A : B :: C : I^2 - - - - - 207
But A : B :: C : D - - - - - theo.
Th. C : I^2 :: C : D - - - - - 173
Th. I^2 is equal to D - - - - 167
Th. the lines H and I are equal - - - 105
Th. E : F :: G : H - - - - - 56

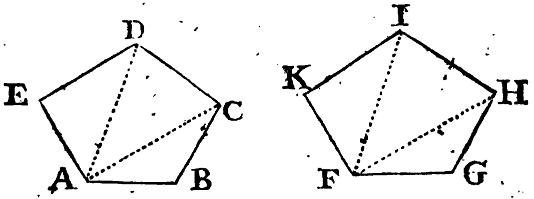
209. *If three ſtrait lines are proportional, the ſquare of the firſt is to the ſquare of the ſecond, as the firſt line to the third.*



Let A, B, C be proportional lines, and $A : B :: B : C$; then $A^2 : B^2 :: A : C$.

For	$A : B :: A^2 : B \times A$	184
And	$B : C :: B^2 : C \times B$	184
But	$A : B :: B : C$	theo.
Th.	$A^2 : B \times A :: B^2 : C \times B$	173
Th.	$A^2 : B^2 :: B \times A : C \times B$	175
Ag.	$A : C :: B \times A : C \times B$	184
Th.	$A^2 : B^2 :: A : C$	173

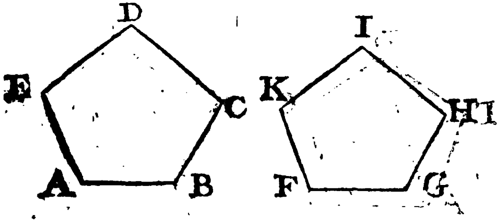
210. *Similar polygons are diviſible into equal numbers of triangles, each triangle in the one equiangular to its correſpondent in the other.*



Let the polygons ABCDE, FGHIK be ſimilar, having the angle ABC equal to the angle FGH, the angle BCD equal to the angle GHI, &c. and join AC, AD, FH, FI; then the triangle ABC is equiangular to the triangle FGH, the triangle ACD is equiangular to FHI, &c.

For the angle ABC is equal to FGH - - 165
And AB : BC :: FG : GH - - - - 165
Th. the triangles ABC, FGH are equiangular - 198
And the angles ACB, FHG are equal - - 198
Th. the angles ACD, FHI are equal - - 50
Ag. BC : GH :: AC : FH - - - - 193
And BC : GH :: CD : HI - - - - 165
Th. AC : FH :: CD : HI - - - - 173
Th. the triangles ACD, FHI are equiangular - 198
By the ſame reaſon ADE, FIK are equiangular, &c. - -

211. *The circuits of ſimilar polygons are proportional to their correſponding ſides.*



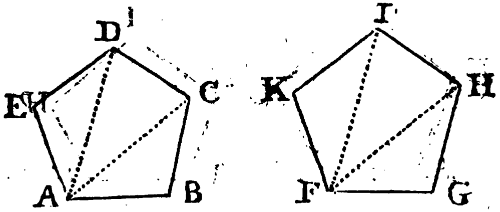
Let the polygons ABCDE, FGHIK be ſimilar, having the angle ABC equal to the angle FGH, the angle BCD equal to the angle GHI, &c; then AB : FG :: the circuit of ABCDE : circuit of FGHIK.

For AB : FG :: BC : GH :: CD : HI, &c. - 165
Th. AB : FG :: AB + BC + CD, &c : FG + GH + HI, &c. - - - - - - - - 179
But AB + BC + CD, &c. is the circuit of ABCDE 52
And FG + GH + HI, &c. is the circuit of FGHIK 52
Th. AB : FG :: circuit of ABCDE : circuit of FGHIK. 56

Remark.

In ſimilar figures, the correſponding parts of their circuits are proportional; for AB : FG :: ABC : FGH :: BCD : GHI, &c. - - - - - 179

212. *The areas of ſimilar polygons are proportional to the ſquares of their correſponding ſides.*

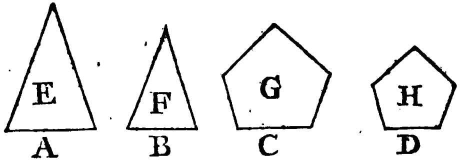


Let the polygons ABCDE, FGHIK be ſimilar, and have the angle ABC equal to the angle FGH, the angle BCD equal to GHI, &c. then $AB^2 : FG^2 :: ABCDE : FGHIK$.

For join AC, AD, FH, FI.

Now the triangles in one polygon are equiangular to thoſe in the other, each to its correſpondent 210
Th. $AB : FG :: AC : FH :: AD : FI$ - - 193
Th. $AB^2 : FG^2 :: AC^2 : FH^2 :: AD^2 : FI^2$ - - 207
But $AB^2 : FG^2 :: ABC : FGH$, &c. - - 205
Th. $ABC : FGH :: ACD : FHI :: ADE : FIK$ - - 173
Th. $ABC : FGH :: ABCDE : FGHIK$ - - 179
Th. $AB^2 : FG^2 :: ABCDE : FGHIK$ - - 173

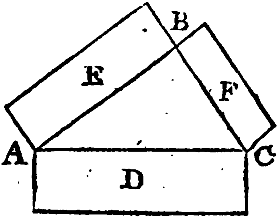
213. *If four ſtrait lines are proportional, figures deſcribed upon them ſimilar and alike ſituated, will alſo be proportional.*



If the ſtrait lines A, B, C, D are proportional, A : B :: C : D, and upon A, B the figures E, F be deſcribed ſimilar to each other alike ſituated, and alſo upon C, D the figures G, H be deſcribed ſimilar to each other and alike ſituated; I ſay, E : F :: G : H.

For $A^2 : B^2 :: C^2 : D^2$ - - - - 207
And $A^2 : B^2 :: E : F$ - - - - 212
Th. $C^2 : D^2 :: E : F$ - - - - 173
Ag. $C^2 : D^2 :: G : H$ - - - - 212
Th. $E : F :: G : H$ - - - - 173

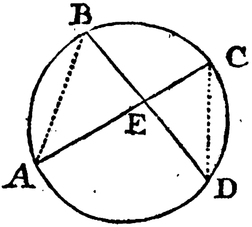
214. *In right angled triangles, any figure deſcribed upon the ſide oppoſite to the right angle, will be equal to both the figures deſcribed upon the ſides containing it, if all are ſimilar and alike ſituated.*



If the triangle ABC is right angled at B, and the figures D, E, F deſcribed on its ſides be ſimilar to one another and alike ſituated; then D is equal to both E and F.

For $AB^2 : AC^2 :: E : D$ - - - - - 212
Th. $AB^2 : E :: AC^2 : D$ - - - 175
Ag. $AB^2 : BC^2 :: E : F$ - - - - 212
Th. $AB^2 : AB^2 + BC^2 :: E : E + F$ - - 176
Th. $AB^2 : E :: AB^2 + BC^2 : E + F$ - - 175
Th. $AC : D :: AB^2 + BC^2 : E + F$ - - - 173
But $AC^2 = AB^2 + BC^2$ - - - - 100
Th. $D = E + F$ - - - - - - 167

215. *If two ſtrait lines within a circle cut each other and terminate in the circumference, the rectangle contained by the parts of the one, will be equal to the rectangle contained by the parts of the other.*



In the circle ABCD, let the ſtrait lines AC, BD cut each other in E; then AE × EC = BE × ED.

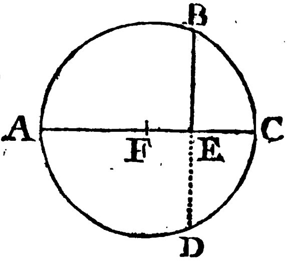
For join AB, CD.

Now the angle AEB = CED - - - 70
- - - the angle ABD = ACD - - - 143
And the angle BAC = BDC - - - 143
Th. the triangles AEB, DEC are equiangular - 27
Th. AE : EB :: DE : EC - - - 193
Th. AE × EC = EB × DE - - - - 185
Or AE × EC = BE × ED - - - 185

216. Corol-

216. Corollary.

In a circle, the ſquare of a line perpendicular to any diameter, is equal to a rectangle contained by the ſegments of that diameter.

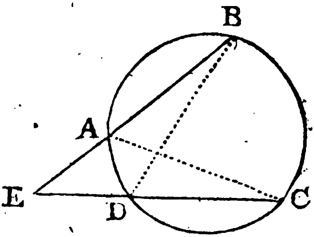


In the circle ABCD, let the line BE be drawn perpendicular to the diameter AC; then $BE^2 = AE \times EC$.

For produce BE to D.

Now	AEB is a right angle - - -	theo.
Th.	AED is a right angle - - - - -	67
Th.	BE and ED are equal - - -	127
But	$BE \times ED = AE \times EC$ - - -	215
Th.	$BE \times BE = AE \times EC$ - - -	56
Or	- - - $BE^2 = AE \times EC$ - - -	61

217. *If from any point without a circle, ſtrait lines be drawn cutting it, and terminating in the circumference; the rectangles contained by the whole lines and their external parts, will be equal to one another.*

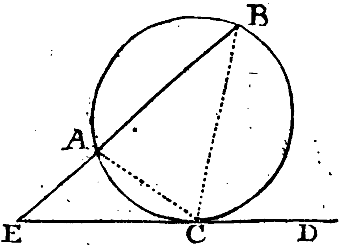


From the point E without the circle ABCD, let the ſtrait lines EB, EC be drawn cutting the circumference in A and D; then $EB \times AE = EC \times DE$.

For join AC, BD.

Now the angle ABD is equal to ACD - - 143
And the angle BEC is common - - - -
Th. the angles CAE, EDB are equal - - 87
Th. the triangles AEC, DEB are equiangular - 27
Th. $AE : EC :: DE : EB$ - - - 193
Th. $AE \times EB = EC \times DE$ - - - - 185
Or $EB \times AE = EC \times DE$ - - - - 185

218. *If from any point without a circle, one line be drawn cutting the circle and another touching it; the rectangle contained by the whole cutting line and its external part, will be equal to the ſquare of the tangent.*

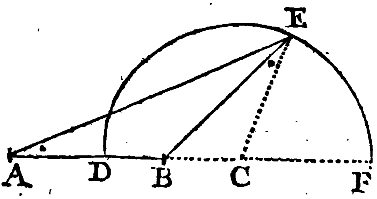


From the point E without the circle ABC, let the ſtrait line EB be drawn cutting the circumference in A, and ED touching it in C; then $EB \times AE = EC^2$.

For join AC, CB.

Now the angle ACE is equal to EBC - - 155
And the angle AEC is common - - - -
Th. the angles CAE, ECB are equal - - 87
Th. the triangles AEC, CEB are equiangular - 27
Th. AE : EC :: CE : EB - - - - 193
Th. $AE \times EB = EC^2$ - - - - - 186
Or $EB \times AE = EC^2$ - - - - - 186

219. *If a ſtrait line is cut by a circle into two ſuch parts, that their difference is to the leſſer, as the greater to the radius of the circle, being the leſſer part produced; theſe parts will be proportional to lines joining any point in the circumference and the ends of the ſaid line.*

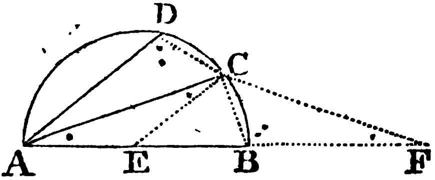


Let the ſtrait line AB be produced to C, and about the center C let the circle DEF be deſcribed cutting AB in D, ſo that AD — DB : DB :: AD : DC, and join AE, EB; then AD : DB :: AE : EB.

For join CE.

Now AD—DB : DB :: AD : DC - - - - theo.
Th. - - - AD : DB :: AC : DC - - - 176
Th. - - - AD : AC :: DB : DC - - - 175
Th. - - - AD : DC :: DB : BC - - - 177
Th. - - - AC : DC :: DC : BC - - - 176
But CE is equal to DC - - - - - 30
Th. AC : CE :: EC : CB - - - - 56
And the contained angle ACE is common - - -
Th. the triangles ACE, ECB are equiangular - 198
Th. - - - AE : EB :: AC : EC - - - 93
Th. - - - AE : EB :: AC : DC - - - - 56
Th. - - - AD : DB :: AE : EB - - - 173

220. *In a circle, if the angle made by a diameter, and a line drawn from its extremity is biſected; the ſquare of the biſecting line, will be equal to a rectangle contained by the ſum of that diameter and line, and the radius of the circle.*



Let ABCD be a ſemicircle, whoſe diameter is AB, and center is E, and from the point A, let the ſtrait line AD be drawn any how, and the line AC biſecting the angle BAD; then $AC^2 = \overline{AB+AD} \times AE$.

For produce AB to F, make BF equal to AD, and join DC, CB, EC, CF.

Now the arch DC is equal to the arch CB - - 148

Th. the line DC is equal to the line CB - - 152

Th. AD, DC are equal to FB, BC each to each - -

And the contained angles ADC, FBC are equal - 45

Th. AC and CF are equal - - - - 62

Th. the angles AFC, CAE, ECA are equal - - 63

Th. the angle AEC is equal to ACF - - 87

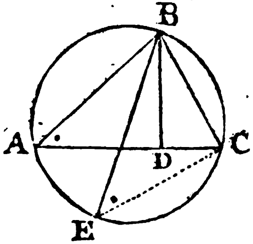
Th. the triangles EAC, CAF are equiangular - 27

Th. EA : AC :: CA : AF - - - - 193

Th. $AC^2 = AF \times AE$ - - - - 185

Th. $AC^2 = \overline{AB+AD} \times AE$ - - - - 56

221. *The rectangle contained by the ſides of a triangle, is equal to the rectangle contained by a perpendicular to the baſe and diameter of a circumſcribing circle.*

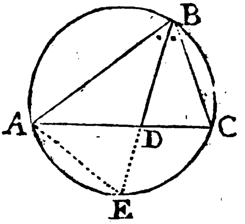


If a circle ABCE is deſcribed about the triangle ABC, and the perpendicular BD, and diameter BE of the circle be drawn; then AB × BC = BD × BE.

For join EC.

Now BCE is a right angle - - - - 153
Th. the angles ADB, BCE are equal - - 54
And the angles BAD, BEC are equal - - 143
Th. the angles ABD, EBC are equal - - 87
Th. the triangles ABD, EBC are equiangular - 27
Th. AB : BD :: EB : BC - - - 193
Th. AB × BC = BD × EB - - - - 185
Or AB × BC = BD × BE - - - - 185

222. *The rectangle contained by the ſides of a triangle, is equal to the ſquare of a line biſecting the verticle angle and meeting the baſe, with a rectangle contained by the parts of the baſe.*

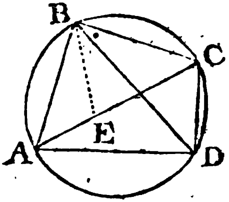


In the triangle ABC, let the line BD be drawn making the angles ABD, DBC equal, and meeting the baſe in D; then AB × BC = BD^2 + AD × DC.

For deſcribe a circle ABCE about the triangle by problem 34, produce BD to E, and join AE.

Now	the angle ABD = DBC	theo.
And	the angle AEB = BCD	143
Th.	the angle BAE = BDC	87
Th.	the triangles ABE, DBC are equiangular	27
Th.	AB : BE :: DB : BC	193
Th.	AB × BC = BE × BD	185
But	BD^2 + DE × BD = BE × BD	108
Th.	AB × BC = BD^2 + DE × BD	48
Ag.	AD × DC = DE × BD	215
Th.	AB × BC = BD^2 + AD × DC	56

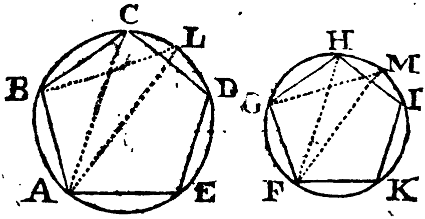
223. *If a quadrangle be inſcribed in a circle, the rectangle contained by its diagonals, will be equal to both the rectangles contained by the oppoſite ſides.*



Let the quadrangle ABCD be inſcribed in a circle, and draw the diagonals AC, BD; then AC × BD = AB × CD + AD × BC.

For make the angle ABE equal to DBC - - p. 9
Now the angle BAE = BDC - - - 143
- - - the angle ADB = ECB - - - 142
And the angle ABD = EBC - - - 49
Th. the triangles BAE, BDC are equiangular, and the triangles ADB, ECB are equiangular - - 87
Th. BA : AE :: BD : DC - - - 192
And AD : DB :: EC : CB - - - 192
Th. AB × CD = AE × BD - - - 185
And AD × BC = EC × BD - - - - 184
Th. AB × CD + AD × BC = AE × BD + EC × BD - 48
But AC × BD - - - - - - - = AE × BD + EC × BD - 106
Th. AC × BD - - - - - - - = AB × CD + AD × BC - 48

224. *The circuits of ſimilar polygons deſcribed in circles are proportional to the diameters, and their areas to the ſquares of the diameters.*



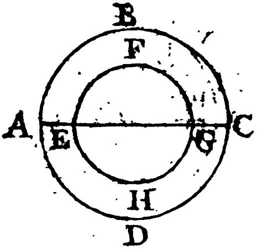
Let ABCDE, FGHIK be ſimilar polygons inſcribed in circles, whoſe diameters are AL, FM; then AL : FM :: circuit of ABCDE : circuit of FGHIK, and $AL^2 : FM^2$ :: area of ABCDE : area of FGHIK.

For join AC, BL, FH, GM.

Now the angle BCA = BLA, and GHF = GMF - 143
Alſo the angle BCA = GHF - - - 210
Th. the angle BLA = (GHF =) GMF - - 48
And ABL, FGM are right angles - - 153
Th. the triangles ABL, FGM are equiangular - 87
Th. AB : FG :: AL : FM - - - - 193
Th. $AB^2 : FG^2 :: AL^2 : FM^2$ - - - 207
But AB : FG :: circuit ABCDE : circuit FGHIK. 211
And $AB^2 : FG^2$ :: area of ABCDE : area of FGHIK. 212
Th. AL : FM :: circuit ABCDE : circuit FGHIK. 173
And $AL^2 : FM^2$ :: area of ABCDE : area of FGHIK. 173

225. Corollary.

The circumferences of circles are proportional to their diameters, and the areas to the ſquares of their diameters.



Thus let ABCD, EFGH be circles, whoſe diameters are AC, EG.

Then AC : EG :: circumf. ABCD : circumf. EFGH. 224

And $AC^2 : EG^2$ :: area of ABCD : area of EFGH. 224

Becauſe, circles may be conſidered as agreeing with ſimilar inſcribed polygons, whoſe ſides are indefinitely ſmall.

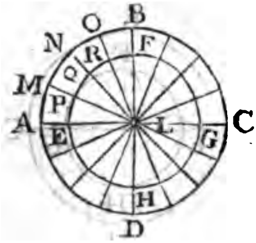
226. Corollary.

If there be three ſuch circles, that the ſquare of one of their diameters, is equal to the ſum of the ſquares of the other two diameters, the greater circle is equal to both the leſſer 214

226. Other-

225. Otherwiſe.

The circumferences of circles are as the ſemidiameters, and the areas are as the ſquares of the ſemidiameters.



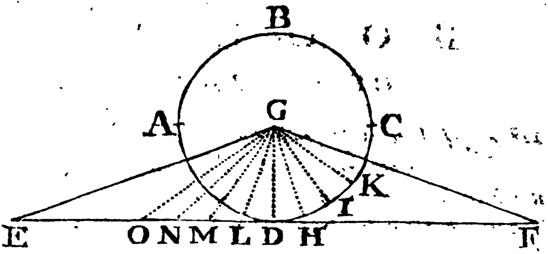
Let the circles ABCD, EFGH have a common center at L, and join AL; then AL : EL :: circumf. ABCD : circumf. EFGH, and $AL^2 : EL^2$ :: area of ABCD : area of EFGH.

For conceive the circumference ABCD divided into equal parts AM, MN, NO, &c. and draw LM, LN, LO, &c. cutting the leſſer circle in E, P, Q, &c.

Now the angles ALM, MLN, NLO, &c. are equal 150
Th. the arches EP, PQ, QR, &c. are equal - 148
Alſo the ſectors ALM, MLN, NLO, &c. are equal 149
And the ſectors ELP, PLQ, QLR, &c. are equal 149
Th. AM : EP :: circumf. ABCD : circumf. EFGH 174
And ALM : ELP :: area of ABCD : area of EFGH 174

Again, let the parts AM, MN, NO, &c. diminiſh, till the ſectors ALM, ELP may be conſidered as equiangular triangles - - - - - - 170
Then AM : EP :: AL : EL - - - 193
And ALM : ELP :: $AL^2 : EL^2$ - - - 205
Th. AL : EL :: circumf. ABCD : circumf. EFGH 173
And $AL^2 : EL^2$ :: area of ABCD : area of EFGH 173

227. *A circle is equal to a triangle, whoſe baſe is equal to the circumference, and altitude is equal to the ſemidiameter.*



Let a circle ABCD, have a tangent EF touching it in D, whoſe length is equal to the circumference, and to the center of the circle G, draw EG, FG; then the circle ABCD, is equal to the triangle EFG.

For conceive the circumference of the circle ABCD, and baſe of the triangle EF, divided into parts DH, HI, IK; DL, LM, MN, &c. all equal to one another, indefinitely ſmall (170) and join each point of diviſion to G the center of the circle.

Now the angles DGH, HGI, IGK, &c. are equal 159
Th. the ſectors DGH, HGI, IGK, &c. are equal 149
Alſo the triangles DGL, LGM, MGN, &c. are equal. 96
And the numbers of each are equal - - con.
Th. DGH : DGL :: circle ABCD : triangle EFG - 174
But the baſes DH, DL being equal, and indefinitely ſmall, the ſector DGH, muſt be equal to the triangle DGL - - - - - - - - 96
Th. the circle ABCD, is equal to the triangle EFG. 159

BOOK

BOOK V.

OF SOLIDS.

228. *A Line is perpendicular to a plain, when it makes right angles with all lines meeting it which are drawn in the ſaid plain.*

229. *A plain is perpendicular to a plain, when all lines in the one drawn at right angles to their common ſection, are perpendicular to the other.*

230. *The inclination of a line to a plain, is the angle contained by it, and a line joining the points of ſection, made by the ſaid line, and a perpendicular falling on the plain from its extremity.*

231. *The inclination of a plain to a plain, is an acute angle contained by lines drawn in each plain, from any point in their common ſection at right angles to it.*

232. *Parallel plains, are thoſe which if infinitely produced, will never meet.*

233. *A pyramid is contained by ſeveral plains ſtanding upon a plain, and meeting in one point.*

234. *A priſm is contained by ſeveral plains, the two baſes of which are equal, ſimilar and parallel; but the others are parallelograms.*

235. *A cube is contained by ſix equal ſquares.*

236. *A ſphere or globe is generated by the revolution of a ſemicircle about its diameter, called the axis.*

237. *The center of a globe is the ſame as the center of the generating ſemicircle.*

238. *The diameter of a globe, is a line paſſing thro' the center, and terminating in the ſurface.*

239. *A cone is generated by the revolution of a ſtrait line about the circumference of a circle, and always paſſing thro' a fixed point above the plain of the ſaid circle.*

240. *The axis of a cone is the ſtrait line joining the fixed point and center of the circle.*

241. *A cylinder is generated by the revolution of a ſtrait line, about the circumferences of two parallel and equal circles, and always parallel to a ſtrait line joining their centers.*

242. *The axis of a cylinder, is the ſtrait line joining the centers of the parallel circles.*

243. *Right cones and cylinders, are thoſe whoſe axes are perpendicular to their baſes.*

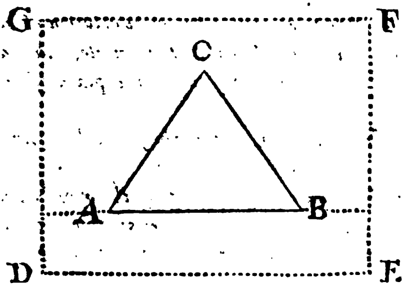
244. *Right cones and cylinders are ſimilar, when their axes are proportional to the diameters of their baſes.*

245. *Solids are ſimilar, which are contained by equal numbers of ſimilar plains, alike ſituated.*

246. An Axiom.

If ſolids of equal baſe and altitude ſtanding upon the ſame plain, are cut by plains parallel to their baſes, and all the correſponding ſections poſſible be equal to each other, thoſe ſolids fill equal ſpaces, and are equal to one another.

247. *The three ſides of a triangle, are in the ſame plain.*



Let ABC be a plain triangle, then its ſides AB, BC, CA are in the ſame plain.

For conceive a plain DEFG paſſing through the points A and B, to revolve, or turn about the line AB as an axis, till it falls upon the point C, and there remains at reſt.

Now the points A, B, C are in the plain DEFG. - above

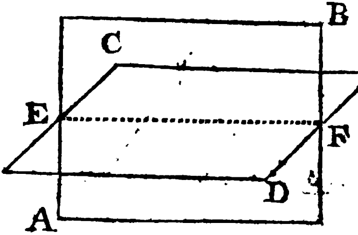
Therefore, the lines AB, BC, CA are alſo in the plain DEFG - - - - - - - - 6

Conſequently, the three ſides of any plain triangle, are in one plain.

248. Corollary.

A plain will paſs through any two ſtrait lines which meet each other.

249. *The common ſection of two plains, is a ſtrait line.*



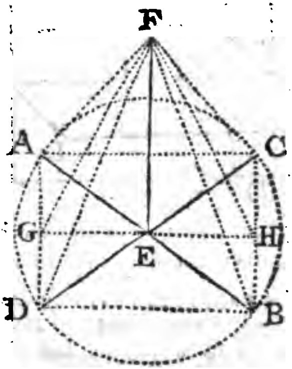
Let the plains AB, CD cut one another in E and F, and join EF; then the ſtrait line EF is their common ſection.

For the points E and F being in the plain AB, the ſtrait line EF will agree with the ſaid plain - - 6
By the ſame reaſon, the line EF will agree with the plain CD - - - - - - - - -
Th. the ſtrait line EF muſt be their common ſection.

Remark.

If many plains paſs thro' two points E and F, the ſtrait line EF is the common ſection of them all.

250. *If a ſtrait line ſtands at right angles to two ſtrait lines cutting one another, at the point of ſection, it will be perpendicular to the plain in which thoſe lines are ſituate.*



Let the line EF ſtand at right angles to the lines AB, CD cutting each other in E; then EF is perpendicular to the plain ACBD.

For take EA, EC, EB, ED equal to one another, join AD, CB, draw GEH at pleaſure meeting AD, CB in G and H, and join FA, FG, FD, FB, FH, FC.

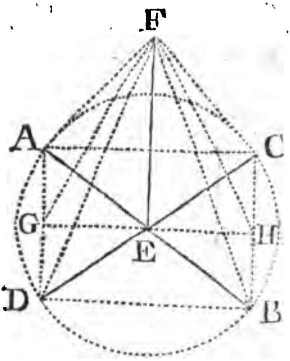
Now EA, ED, EB, EC are equal - - - - con.

And the contained angles AED, BEC are equal - 70

Th. the angle GAE = EBH, and AD = BC - - 62

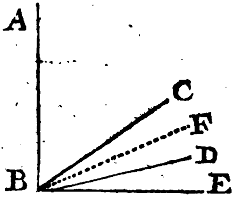
But the angle AEG = BEH - - - - 70

Therefore - - - AG = BH, and GE = EH - - 77



2. The angles FEA, FED, FEB, FEC are equal - 54
Th. FA, FD, FB, FC are equal - - - 62
Th. the angles FAD, FBC are equal - - 66
3. The ſides FA, AG are equal to FB, BH each to each - - - - - - - above
And the contained angles FAG, FBH are equal. above
Therefore, FG is equal to FH - - - 62
Laſtly, FG, GE, EF are equal to FH, HE, EF each to each - - - - - - - above
Th. the angles GEF, FEH are equal - - 66
Th. GEF, FEH are right angles - - - 9
Th. EF is perpendicular to the plain ACBD - 228

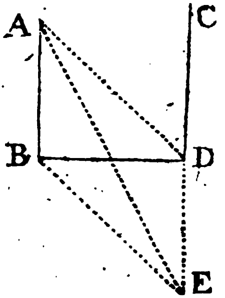
251. *If a ſtrait line ſtands at right angles to three ſtrait lines meeting one another in their common ſection, thoſe three lines are in the ſame plain.*



Let the line AB ſtand at right angles to each of the ſtrait lines BC, BD, BE meeting in B; then thoſe lines are in the ſame plain.

For if not, let BF be the common ſection of the plains paſſing thro' AB, BC and EB, BD - - 248
Now BF is a right line - - - - - 249
And AB is perpendicular to the plain paſſing thro' EB, BD - - - - - - - 250
Th. ABF is a right angle - - - - 228
But ABC is a right angle - - - theo.
Th. the angles ABF, ABC are equal - - 54
Which is impoſſible - - - - - 52
Th. BC, BD, BE muſt be in the ſame plain - -

252. *Two ſtrait lines perpendicular to the ſame plain, are parallels.*

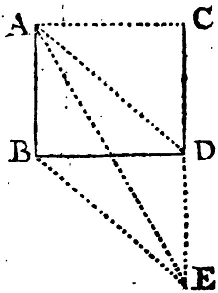


Let the lines AB, CD ſtand at right angles to the plain BD; then AB, CD are parallels.

For join BD, draw DE at right angles to BD in the given plain, make DE equal to AB, and join BE, EA, AD.

Now AB, BD are equal to ED, DB each to each - con.
And the contained angles ABD, EDB are equal - 54
Th. AD is equal to BE - - - - - 62
Ag. AB, BE are equal to ED, DA each to each. above
And AE is common - - - - - - [illegible]
Th. the angles ABE, EDA are equal - - - 66
But ABE is a right angle - - - - - 228
Th. EDA is a right angle - - - - - 54
Laſtly, EDC is a right angle - - - - 228
Th. DC is in the ſame plain with DA, DB - - 251
Alſo AB is in the ſame plain with DA, DB - - 247
Th. AB, CD are parallels - - - - - 81

253. *If two ſtrait lines be parallel, and one of them ſtands at right angles to a plain, the other will alſo ſtand at right angles to the ſame plain.*

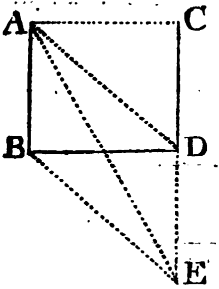


Let the lines AB, CD be parallels, and let AB ſtand at right angles to the plain BD; then CD will alſo ſtand at right angles to that plain.

For join BD, draw DE at right angles to BD in the given plain, make DE equal to AB, and join BE, EA, AD.

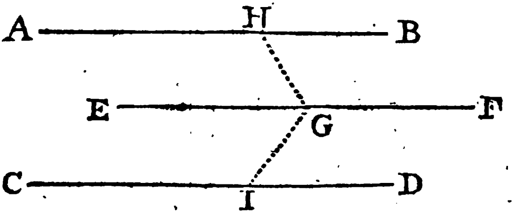
Now the ſides AB, BD are equal to ED, DB each to each - - - - - - - - con.
And the contained angles ABD, EDB are equal - 54
T[illegible] AD is equal to BE - - - - 62

Ag.



Ag. the ſides AB, BE are equal to ED, DA each to each - - - - - - - - above
And the ſide AE is common - - - - - -
Th. the angles ABE, EDA are equal - - 66
But ABE is a right angle - - - - - 228
Th. BDA is a right angle - - - - 54
Th. DE is perpendicular to the plain ABDC, paſſing thro' BD, DA - - - - - - - 250
Th. CDE is a right angle - - - - 228
Alſo CDB is a right angle - - - - 84
Th. CD ſtands at right angles to the given plain, which paſſes thro' BD, DE - - - - 250

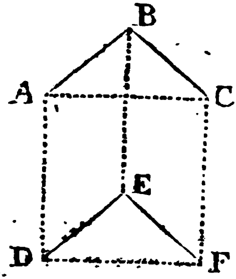
254. *Lines which are parallel to the ſame line, tho' not in the ſame plain with it, are yet parallel to one another.*



Let the lines AB, CD be parallel to the line EF, but not in the ſame plain with it; I ſay, AB, CD are parallels.

For let GH, GI be drawn at right angles to EF, and meeting AB, CD in H and I, by problem 6, and let a plain be drawn thro' HG, GI - - - - 248
Now EF is at right angles to the plain paſſing through HG, GI - - - - - - - - 250
Th. AB, CD are at right angles to that plain - 253
Th. AB, CD are parallels - - - - 252

255. *If two lines meeting each other, be parallel to two other lines meeting each other, tho' not in the ſame plain, yet thoſe lines will contain equal angles.*

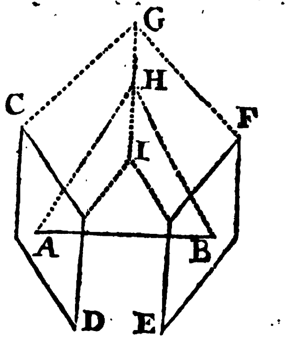


Let the lines AB, BC be parallel to DE, EF, but not in the ſame plain; I ſay, the angles ABC, DEF are equal.

For take BA, BC, ED, EF equal to one another, and join AC, DF, AD, BE, CF.

Now both AD and CF, are parallel to BE and equal to it - - - - - - - - - - - 90
Th. AD, CF are equal - - - - - 48
And AD, CF are parallel - - - - - 254
Th. AC is equal to DF - - - - - 90
Th. AB, BC, CA, are equal to DE, EF, FD each to each. - - - - - - - - - -
Th. the angle ABC is equal to DEF - - 66

256. *Plains to which the ſame ſtrait line is perpendicular, are parallel to one another.*

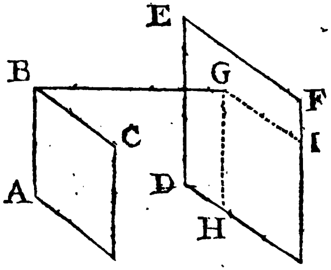


Let the ſtrait line AB, be perpendicular to the plains DC, EF; then the plains DC, EF are parallel.

For if not, let them be produced and meet in GHI, and join AH, HB.

Now ABH is a right angle - - - - 228
And BAH is a right angle - - - - 228
Therefore the triangle ABH has two right angles - -
Which is impoſſible - - - - - - 86
Th. the plains DC, EF produced, will never meet - -
Th. the plains DC, EF are parallel - - 231

257. *If two lines meeting each other, be parallel to two other lines meeting each other, but not in the ſame plain, the plains which paſs through them will be parallel.*

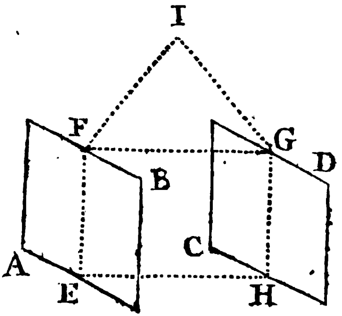


Let the lines AB, BC be parallel to DE, EF not in the ſame plain; I ſay, the plains paſſing thro' AB, BC and DE, EF are parallel.

For let BG be drawn perpendicular to the plain DF, by problem 46, and GH, GI parallel to ED, EF, by problem 10.

Now BGH is a right angle - - - 228
And BA is parallel to GH - - - 254
Th. GBA is a right angle - - - - 84
By the ſame reaſon GBC is a right angle - - [illegible]
Th. BG is perpendicular to the plain AC - - 250
Th. the plains AC, DF are parallel - - 256

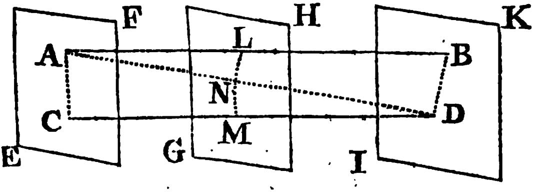
258. *If a plain cuts two parallel plains, their common ſections will be parallel.*



Let the parallel plains AB, CD be cut by the plain EFGH, making the ſections EF, GH; then the lines EF, GH are parallels.

For if not, let them be produced and meet in I. 59
Now EF, HG are ſtrait lines - - - - 249
And each line agrees with its plain - - 6
Th. the plains AB, CD being produced, will meet in I.
Which is impoſſible - - - - - 232
Th. the lines EF, HG being produced, will never meet.
Th. EF, GH are parallels - - - - 12

259. *If two ſtrait lines are cut by parallel plains, they will be cut proportionally.*

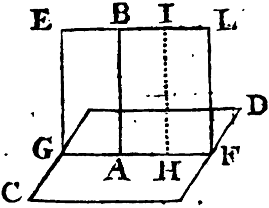


Let the ſtrait lines AB, CD be cut by parallel plains EF, GH, IK in the points A, L, B, C, M, D; then AB : AL :: CD : CM.

For join AC, AD, DB, let AD cut the plain GH in N, and draw LN, NM.

Now the triangle ABD lies in a plain - - 247
Th. LN and BD are parallel - - - 258
By the ſame reaſon, NM and AC are parallel - -
Th. AN : ND :: AL : LB - - - - 190
And AN : ND :: CM : MD - - - - 190
Th. AL : LB :: CM : MD - - - - 178
Th. AB : AL :: CD : CM - - - - 176

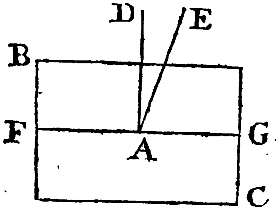
260. *If a ſtrait line ſtands at right angles to any plain, all the plains which paſs thro' it will be perpendicular to the ſame plain.*



Let the line AB ſtand at right angles to the plain CD, and let the plain EF paſs thro' the ſaid line AB; then the plain EF is perpendicular to the plain CD.

For let FG be the common ſection of thoſe plains, in which aſſume the point H, and in the plain EF
ereƈt HI perpendicular to GF. - - - - p. 6
Now BAH is a right angle - - - - theo.
And AHI is a right angle - - - - con.
Th. AB, HI are parallel - - - - - 81
Th. HI is at right angles to the plain CD - 253
Th. the plain EF is perpendicular to the plain CD. 229

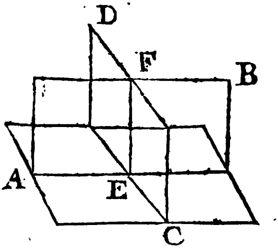
261. *At a given point in a given plain, only one perpendicular can be erected to that plain, on the ſame ſide.*



At the point A, in the plain BC, only one perpendicular can be erected to that plain.

For if poſſible, let both AD and AE be perpendicular to the plain BC, and thro' DA, AE let a plain be drawn, making with BC the ſection FAG. - - 248
Now FAG is a ſtrait line - - - - 249
Th. DAF is a right angle - - - - 228
Alſo EAF is a right angle - - - - 228
Th. the angles DAF, EAF are equal - - 54
Which is impoſſible - - - - - - 52
Conſequently, only one perpendicular can be erected to a given plain, at a given point in it.

262. *If two plains cutting each other, be perpendicular to ſome plain, their common ſection will alſo be perpendicular to that plain.*



Let the plains AB, CD cutting each other, be perpendicular to the plain AE; then their common ſection EF will alſo be perpendicular to the ſame plain.

For let AE, EC be the common ſections made with the plain AE, and at the point E to each of the lines AE, EC, in each of the plains AB, CD erect a perpendicular - - - - - - - - p. 6

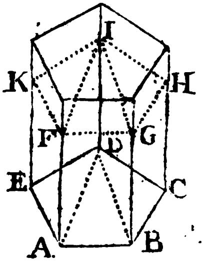
Now the perpendiculars erected to AE, EC in the plains AB, CD, are perpendicular to the plain AE. 229

But two perpendiculars cannot be erected to a plain, at the ſame point, and on the ſame ſide. - 261

Conſequently, both thoſe perpendiculars muſt agree with EF, the common ſection of the plains AB, CD. 249

Th. EF is perpendicular to the plain AE - -

263. *If a priſm be cut by a plain parallel to its baſe, the ſection will be equal to the baſe.*



Let ABCDE be the baſe of a priſm, and FGHIK a ſection parallel to it; then FGHIK is equal to ABCDE.

For join AD, DB, FI, IG.

Now DI is parallel to AF, and to BG - - 254

Th. FI, IG are parallel to AD, DB - - 258

Th. FI, IG are equal to AD, DB - - 94

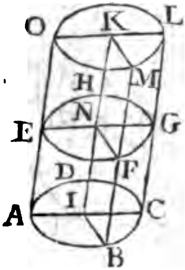
And the contained angles FIG, ADB are equal - 255

Th. the triangle FIG is equal to ADB - - 62

By the ſame reaſon, each triangle in the ſection, is equal to its correſponding triangle in the baſe -

Th. FGHIK is equal to ABCDE - - - 49

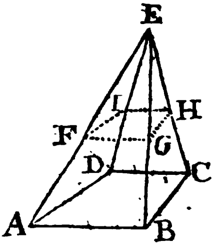
264. *If a cylinder be cut by a plain parallel to its baſe, the ſection will be a circle, equal to the baſe.*



Let ABCD be the baſe of a cylinder AL, and EFGH be a ſection parallel to the baſe; then EFGH will be a circle, and equal to ABCD.

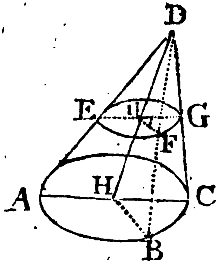
For let the plains IKMB, IKLC cut the cylinder thro' its axis IK, and meet the ſection in N, F, G.
Now BM, CL are parallel to IK - - - 241
And NF, NG are parallel to IB, IC - - 258
Th. NF, NG are equal to IB, IC - - - 91
But IB and IC are equal - - - - 30
Th. NF and NG are equal - - - - 48
By the ſame reaſon, all lines drawn from the point N to the circumference of the ſection, are equal - -
Th. the ſection EFGH is a circle - - - - 30
And the circles EFGH, ABCD are equal - - 55

265. *If a pyramid be cut by a plain parallel to its baſe, the ſection will be ſimilar to the baſe.*



Let ABCDE be a pyramid, and FGHI a ſection parallel to the baſe, then FGHI is ſimilar to ABCD.
For FG, GH are parallel to AB, BC - - 258
Th. EB : EG :: AB : FG - - - - 193
And EB : EG :: BC : GH - - - - 193
Th. AB : FG :: BC : GH - - - - 173
And the angles ABC, FGH are equal - - 255
By the ſame reaſon, each angle in the ſection is equal to its correſponding angle in the baſe, and the ſides about them are proportional - - - -
Th. FGHI is ſimilar to ABCD - - - 165

266. *If a cone be cut by a plain parallel to its baſe, the ſection will be a circle.*

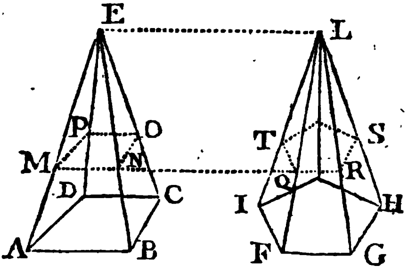


Let ABCD be a cone, and EFG a ſection parallel to the baſe, then EFG is a circle.

For let the plains ADC, BDH cut the cone through its axis DH, and meet the ſection in I, G, F.

Now IF, IG are parallel to HB, HC - - 258
Th. DH : DI :: HB : IF - - - - 193
And DH : DI :: HC : IG - - - - 193
Th. HB : IF :: HC : IG - - - - 173
But HB is equal to HC - - - - 30
Th. IF is equal to IG - - - - 167
By the ſame reaſon, all lines drawn from the point I to the circumference of the ſection, are equal to IG.
Th. the ſection EFG is a circle - - 30

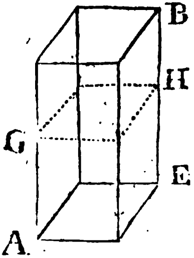
267. *If pyramids or cones of equal altitude ſtanding upon the ſame plain, be cut by a plain parallel to their baſes, the ſections and baſes will be proportional.*

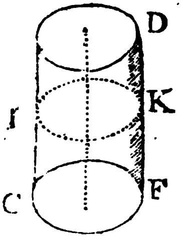


Let the pyramids ABCDE, FGHIL of equal altitude, ſtanding upon the ſame plain, be cut by a plain parallel to their baſes AC, FH, making the ſections MNOP, QRST, then AC : MO :: FH : QS.

For MN, QR are parallel to AB, FG - - 258
Th. EA : EM :: AB : MN - - - - 193
And LF : LQ :: FG : QR - - - - 193
But EA : EM :: LF : LQ - - - - 259
Th. AB : MN :: FG : QR - - - - 173
And the ſections are ſimilar to the baſes - - 265
Th. AC : MO :: FH : QS - - - - 213

268. *Priſms and cylinders, of equal baſe and altitude, are equal to one another.*



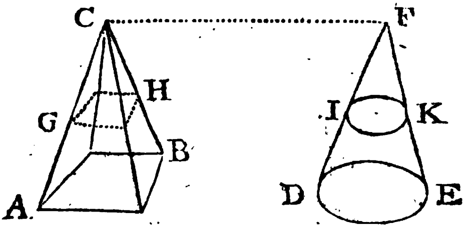


Let the priſm AB and cylinder CD, have equal baſes AE, CF, and alſo equal altitudes EB, FD; then the ſolids AB, CD are equal.

For conceive them to ſtand upon the ſame plain, and to be cut by plains parallel to their baſes, making the correſponding ſections GH, IK, &c.

Now the ſection GH is equal to the baſe AE - 263
And the ſection IK is equal to the baſe CF - 264
But the baſes AE, CF are equal - - - theo.
Th. the ſections GH, IK are equal - - - 48
By the ſame reaſon, every ſection in AB is equal to its correſponding ſection in CD - - - - -
Th. the ſolids AB, CD are equal - - 246

269. *Pyramids and cones of equal baſe and altitude, are equal to one another.*

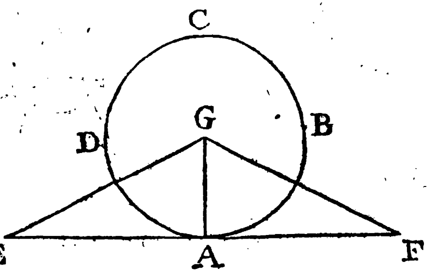


Let the pyramids or cones ABC, DEF have equal baſes AB, DE, and alſo equal altitudes; then the ſolids ABC, DEF are equal.

For let them ſtand upon the ſame plain, and be cut by plains parallel to their baſes, making the correſponding ſections GH, IK, &c.

Now AB : GH :: DE : IK - - - - 267
And AB is equal to DE - - - - theo.
Th. GH is equal to IK - - - - 267
By the ſame reaſon, every ſection made in ABC, is equal to the correſponding ſection made in DEF -
Th. the ſolids ABC, DEF are equal - - 246

270. *A globe is equal to a pyramid or cone, whoſe baſe is equal to its ſurface, and altitude is equal to its ſemidiameter.*

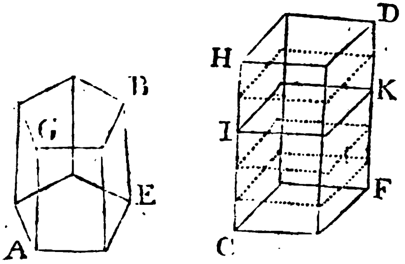


Let the ſurface of the globe ABCD, be equal to the baſe of the pyramid EFG, whoſe altitude is the ſemidiameter of the globe GA; then the globe is equal to the pyramid.

For let the ſurface of the globe and baſe of the pyramid be divided into indefinitely ſmall parts equal to one another, upon which as baſes, let pyramids be conſtituted, whoſe common vertex is in G, the center of the globe - - - - - - - 170.

Now the number of baſes in the ſurface of the globe, are equal to the number in the baſe of the pyramid - - - - - - - con.
And the pyramids conſtituted thereon are equal 269.
Th. the globe ABCD is equal to the pyramid EFG 49.

271. *Priſms and cylinders upon equal baſes, are proportional to their altitudes.*

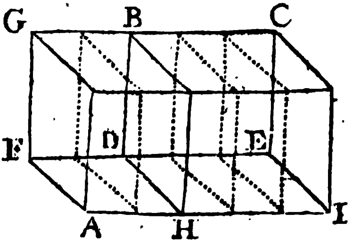


Let AB, CD be priſms or cylinders having equal baſes AE, CF, but unequal altitudes AG, CH; then AG : CH :: AB : CD.

For in CH the greater altitude, take CI equal to AG the leſſer, conceive CI, IH divided into parts equal to one another, and plains paſſing thro' each point of diviſion parallel to the baſe CF.

Now each ſection in CD, is equal to the baſe CF 263
Th. thoſe ſections are equal to one another - - 48
Th. all the little ſolids into which CD is divided by the cutting plains, are equal to one another - 268
Th. CI, CH and CK, CD have equal ratios - - 159
Th. CI : CH :: CK : CD - - - - - 160
But AB is equal to CK - - - - 268
Th. AG : CH :: AB : CD - - - - 56

272. Priſms and cylinders of equal altitude, are proportional to their baſes.

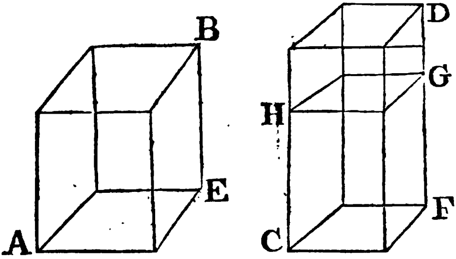


Since all priſms and cylinders of equal baſe and altitude are equal (by 265) let the rectangular priſms AB, AC, whoſe baſes are AD, AE, repreſent any priſms or cylinders of equal altitude BG; then AD : AE :: AB : AC.

For conceive FD, DE divided into parts equal to one another, and plains drawn thro' each point of diviſion parallel to AG or BH.

Now each ſection in the ſolid AC is equal to AG - 263
Th. the ſections are equal to one another - - 48
And the ſections in AE are parallel - - 258
Th. the little rectangles in AD, AE are equal - 94
And the little priſms in AB, AC are equal - - 268
Th. AD, AE and AB, AC have equal ratios - 159
Th. AD : AE :: AB : AC - - - - 160

273. *Priſms and cylinders are equal, whoſe baſes and altitudes are reciprocally proportional.*

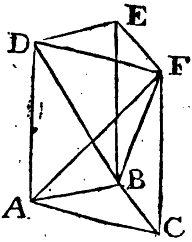


Let the priſms AB, CD have their baſes AE, CF, and altitudes EB, FD reciprocally proportional; namely, AE : CF :: FD : EB; then the ſolids AB, CD are equal.

For in the greater altitude FD, take FG equal to the leſſer EB, and thro' the point G, let a plain be drawn parallel to the baſe of the priſm, making the ſection GH.

Now AE : CF :: FD : EB - - - - theo.
And FG is equal to EB - - - - con.
Th. AE : CF :: FD : FG - - - - 56
But CD : CG :: FD : FG - - - - 271
Th. AE : CF :: CD : CG - - - - 173
Ag. AE : CF :: AB : CG - - - - 272
Th. AB : CG :: CD : CG - - - - 173
Th. AB is equal to CD - - - - 167

274. *Every pyramid that has a triangular baſe, is a third part of a priſm having the ſame baſe and altitude.*

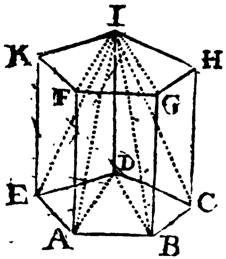


Let the triangle ABC be the baſe of a pyramid ABCF, and likewiſe of a priſm, having the ſame altitude; then the pyramid is a third part of the priſm.

For let DEF be the oppoſite baſe of the priſm, and join AF, FB, BD.

Now the baſe ACF is equal to the baſe ADF - - 91
And the vertex B is common - - - -
Th. the pyramids ACFB, ADFB are equal - - 269
Again, the baſe ADB is equal to the baſe BDE - 91
And the vertex F is common - - - - -
Th. the pyramids ADBF, BDEF are equal - - 269
But the pyramid ADBF is the ſame with ADFB, being contained by the very ſame plains - - - -
Th. the pyramids ACFB, ADFB, BDEF are equal 48
And the priſm ABCDEF is equal to thoſe three pyramids - - - - - - - 52
Th. ACFB, or ABCF is one third of ABCDEF - -

275. Every pyramid is a third part of a priſm, having the ſame baſe and altitude.



Let the figure ABCDE be the common baſe of a pyramid ABCDEI, and a priſm AHIK; then the pyramid is a third part of the priſm.

For join AD, DB, FI, IG.

Now upon each triangle in the baſe ABCDE, is conſtituted a pyramid and a priſm, of the ſame altitude - - - - - - - - con.

And each of thoſe pyramids is a third part of each priſm, having the ſame baſe - - - 274

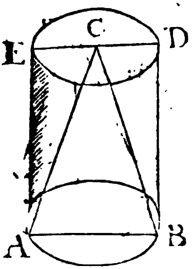
Therefore, all the pyramids taken together, are a third part of all the priſms taken together - - -

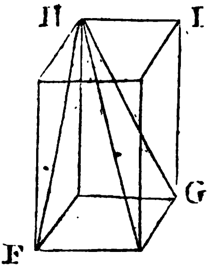
But all thoſe pyramids compoſe the pyramid ABCDEI, and all theſe priſms compoſe the priſm AHIK - -

Th. ABCDEI is one third of AHIK - - 56

276. Every

276. *Every cone is a third part of a cylinder, having the ſame baſe and altitude.*



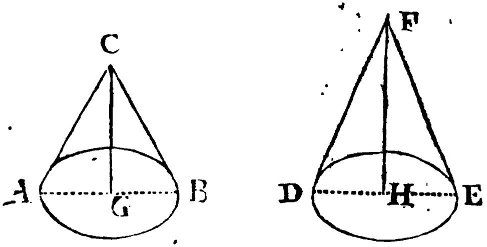


If there be a cone ABC, and a cylinder ABDE having the ſame baſe and altitude; then the cone is one third of the cylinder.

For let there be a pyramid FGH, and a priſm FGHI, having an equal baſe and altitude with the cone and cylinder - - - - - - - - -

Now the priſm FI, is equal to the cylinder AD - - 268
And the pyramid FGH, is equal to the cone ABC. 269
But the pyramid FGH, is one third of the priſm FI - - - - : - - - - 275
Th. the cone ABC, is one third of the cylinder AD. 56

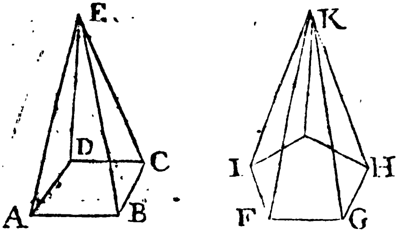
277. *Pyramids and cones upon equal baſes, are proportional to their altitudes.*



Let ABC, DEF be pyramids or cones, with equal baſes, but unequal altitudes CG, FH; then CG : FH :: ABC : DEF.

For priſms or cylinders having equal baſes, are proportional to their altitudes - - - 271
Namely, CG : FH :: cyl. : cyl. - - - 271
But quantities and their third parts are proportional. 174
And pyramids or cones ABC, DEF are the third parts of priſms or cylinders, of equal baſe and altitude. 276
Th. CG : FH :: ABC : DEF - - - - 173

278. *Pyramids and cones of equal altitude, are proportional to their baſes.*



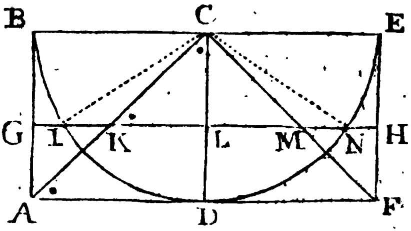
Let ABCDE, FGHIK be pyramids or cones, with equal altitudes, but unequal baſes AC, FH; then as the baſe AC to the baſe FH, ſo is the ſolid E to the ſolid K.

For priſms or cylinders of equal altitude, are proportional to their baſes - - - - - 272
Namely, AC : FH : priſm : priſm - - 272
But quantities and their third parts are proportional. 174
And pyramids or cones E, K, are the third parts of priſms or cylinders, of equal baſe and altitude - 276
Th. AC : FH :: ABCDE : FGHIK - - 173

279. Corollary.

Pyramids and cones are equal, whoſe baſes and altitudes are reciprocally proportional - - 273

180. *A hemiſphere, is two thirds of its circumſcribing cylinder.*



Let the ſquare ABCD, the triangle ACD, and the quadrant BCD, revolve about CD as an axis, and generate the cylinder ABEF, the cone AFC, and the hemiſphere BDE; then $BDE = \frac{2}{3}ABEF$.

For let GH be a ſection parallel to the baſe of the cylinder, and AE be a ſection thro' its axis, cutting the former in I, K, L, M, N.

Now it is evident by the conſtruction of the figure, that CI, CD, DA, GL, LH are equal - - - -

Alſo, that $IL = LN$, $KL = LM$, $KL = LC$, and the angles at the point L are right angles - - -

Th. $CI^2 = IL^2 + LC^2$ - - - - 100

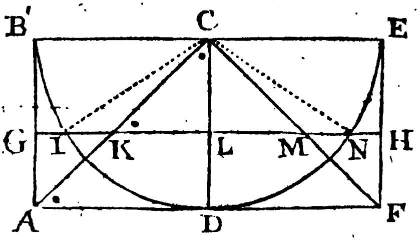
Th. $GL^2 = IL^2 + KL^2$ - - - - 56

Th. $4GL^2 = 4IL^2 + 4KL^2$ - - - - 49

Or $GH^2 = IN^2 + KM^2$ - - - - 110

Th. circle GH = circle IN + circle KM - - 226

181. Th.



Th. the ring GI, NH round the hemiſphere, is equal to the circle whoſe diameter is KM - - 50

By the ſame reaſon, every ſection made in the cup like ſolid ABIDNEF, is equal to its correſponding ſection in the cone AFC.

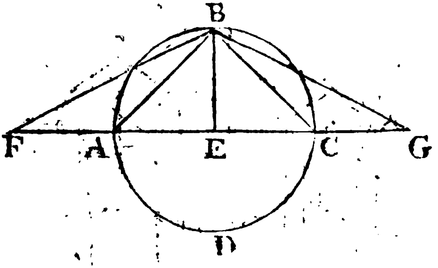
Th. ABIDNEF is equal to the cone AFC - - 246
But the cone AFC is one third of the cylinder AFEB. 276
Therefore, the ſolid ABIDNEF, is one third of the cylinder AFEB - - - - - - - 56
Th. the hemiſphere BDE = $\frac{2}{3}$ AFEB - - 50

Corollary.

281. *A cone, globe, and cylinder circumſcribing them, are proportional to the numbers* 1, 2, 3.

282. *A globe is four times a cone, whoſe baſe is its great circle, and altitude its ſemidiameter.*

283. *The ſurface of a globe is four times the area of its great circle.*

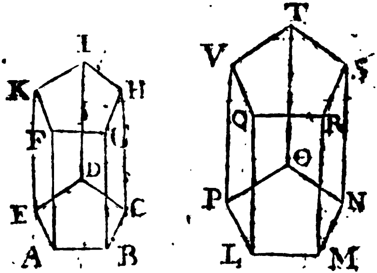


Let the globe ABCD be cut by the plains ABCD, FACG paſſing through the center E, perpendicular to each other, whoſe common ſection AC is a diameter of the globe; then the ſurface of the globe is four times the circle whoſe diameter is AC.

For let ACB, FGB be cones whoſe common altitude is the ſemidiameter EB, and let the baſe FG of the greater cone be equal to the ſurface of the globe.

Now the globe is equal to the cone FGB - - 279
And the globe is four times the cone ACB - 282
Th. the cone FGB is four times the cone ACB - 56
Th. the baſe FG is four times the baſe AC - 278
But the baſe FG is equal to the ſurface of the globe. con.
Th. the ſurface of the globe is four times the area of its great circle, whoſe diameter is AC - - 56

284. *The ſurfaces of ſimilar priſms are proportional to the ſquares of their correſponding lines.*



Let AH, LS be ſimilar priſms, whoſe baſes are AC, LN, and AB, LM correſponding lines; then $AB^2 : LM^2 ::$ ſurface of AH : ſurface of LS.

For AB : LM :: BC : MN :: CD : NO, &c. - 165

Th. $AB^2 : LM^2 :: BC^2 : MN^2 :: CD^2 : NO^2$, &c. - 207

But $AB^2 : LM^2 ::$ AC : LN :: AG : LR, &c. - 212

Th. AC : LN :: AG : LR :: BH : MS, &c. - 173

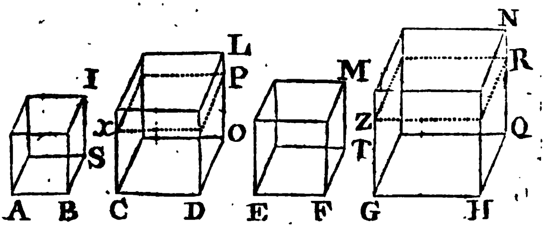
Th. AC : LN :: ſurf. AH : ſurf. LS - - 179

Th. $AB^2 : LM^2 ::$ ſurf. AH : ſurf. LS - - 173

285. Corollary.

The ſurfaces of all ſimilar ſolids are as the ſquares of their correſponding lines, being compoſed of equal numbers, of ſimilar parts.

286. *If ſtrait lines are proportional, their cubes will alſo be proportional.*

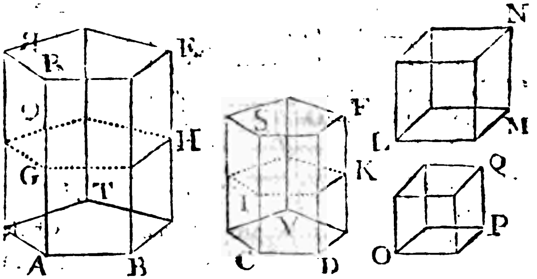


Let AB, CD, EF, GH be proportional lines, whoſe cubes are AI, CL, EM, GN, and AB : CD :: EF : GH; then AI : CL :: EM : GN.

For in OL, QN take OP, QR equal to AB, EF or ST, TM, each to each, and let PX, RZ be ſections parallel to the baſes OC, QG.

Now AB : CD :: EF : GH - - - - theo.
Th. AS : CO :: ET : GQ - - - - 207
But AS : CO :: AI : CP - - - - 272
And ET : GQ :: EM : GR - - - - 272
Th. AI : EM :: CP : GR * - - - - 175
Ag. OP : OL :: CP : CL - - - - 271
And QR : QN :: GR : GN - - - - 271
But OP : OL :: QR : QN, by article 56, being equal to AB, CD, EF, GH each to each - - con.
Th. CP : GR :: CL : GN * - - - 175
Th. AI : EM :: CL : GN - - - - 173
Th. AI : CL :: EM : GN - - - - 175

287. *Similar priſms are proportional to the cubes of their correſponding lines.*



Let AE, CF be ſimilar priſms, whoſe baſes are AT, CV, and AB, CD correſponding lines; then, as cube of AB : cube of CD :: AE : CF.

For make AG, CI equal to AB, CD, let GH, IK be ſections parallel to the baſes, and LN, OQ be the cubes of AB, CD.

Now AR : AG or AB :: AE : AH - - - 271
And CS : CI or CD :: CF : CK - - - 271
But AR : AB - - - :: CS : CD - - 245
Then AE : CF - - - :: AH : CK - - 175
Again LM or AB^2: AT :: LN : AH - - - 272
And OP or CD^2: CV :: OQ : CK - - - 272
But AB^2: AT - - - :: CD^2: CV - - 212
Then LN : OQ - - - :: AH : CK - - 175
Then LN : OQ - - - :: AE : CF - - 173
Or cube of AB : cube of CD :: AE : CF - - -

288. Corol-

Corollaries.

288. *Similar cylinders are proportional to the cubes of their axes, or diameters of their baſes; being of the ſame magnitude with priſms of equal baſe and altitude.* - - - - - - - - 268

289. *Similar pyramids and cones are proportional to the cubes of their correſponding lines; being one third of priſms, or cylinders, that have equal baſes and altitudes.* - - - - - - - - 276

290. *The ſurfaces of globes are proportional to the ſquares of their diameters; being four times the areas of their great circles.* - - - - - 283

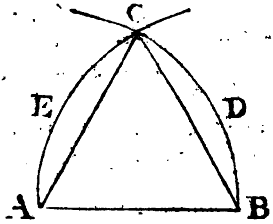
291. *The magnitudes of globes are proportional to the cubes of their diameters; being two thirds of their circumſcribing cylinders.* - - - - 281

292. *All ſimilar ſolids are as the cubes of their correſponding lines, becauſe they are compoſed of equal numbers of ſimilar parts.*

BOOK VI.

Problem I.

293. *Upon a given ſtrait line* AB, *to make an equilateral triangle.*



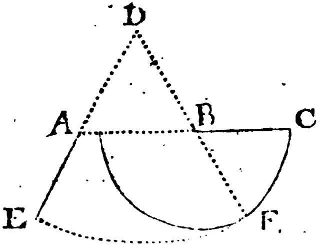
About the center A, with the radius AB, deſcribe the circle BDC, by article 60, alſo about the center B, with the radius BA, deſcribe the circle ABC, and join AC, CB.

For AC is equal to AB - - - - 30
And BC is equal to BA - - - - 30
Th. AC is equal to BC - - - - 48
Th. AB, BC, CA are equal - - - - 48
Th. the triangle ABC is equilateral - - 15

Problem

Problem 2.

294. *From a given point* A, *to draw a line that ſhall be equal to a given ſtrait line* BC.

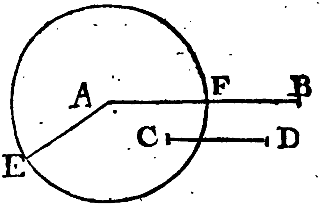


Join AB, and upon it make the equilateral triangle ABD, produce the ſides DA, DB, to E and F, about the center B, with the radius BC deſcribe the arch CF, and about the center D, with the radius DF, deſcribe the arch FE.

For DE is equal to DF - - - - 30
And DA is equal to DB - - - - con.
Th. AE is equal to BF - - - - 50
But BC is equal to BF - - - - - 30
Th. AE is equal to BC - - - - 48

Problem 3.

295. *Two unequal ſtrait lines* AB *and* CD *being given, to take a part of the greater that ſhall be equal to the leſſer.*

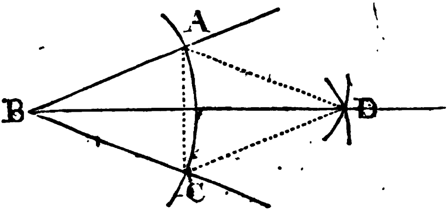


From the point A draw the line AE equal to the line CD, by problem 2, and about the center A, with the radius AE, deſcribe a circle cutting AB in F.

For AE is equal to AF - - - - 30
And AE is equal to CD - - - - con
Th. AF is equal to CD - - - - 48

Problem 4.

296. *To divide a given angle* ABC *into two equal parts, or to biſect it.*



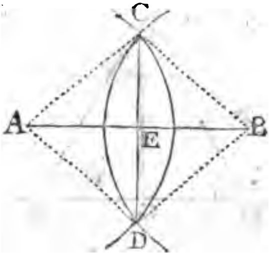
On the center B at any diſtance BA deſcribe the arch AC, on the centers A and C, with the diſtances AB, CB deſcribe arches cutting each other in D, and draw BD.

For join AD, DC, CA.

Now AB, BC, CD, DA are equal - - - con.
Th. the angle BAC = BCA - - - 63
And the angle DAC = DCA - - - 63
Th. the angle BAD = BCD - - - 49
Th. the angle ABD = DBC - - - 62

Problem 5.

297. *To divide a given finite ſtrait line* AB *into two equal parts, or to biſect it.*



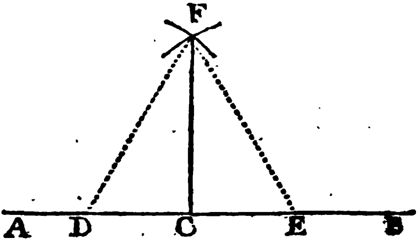
About the centers A and B, with any radius greater than half the line, deſcribe circles cutting one another in C and D, and draw CD cutting AB in E.

For join AC, CB, BD, DA.

Now AC, CB, BD, DA are equal - - - con.
Th. the angle CAB = CBA - - - 63
And the angle DAB = DBA - - - 63
Th. the angle CAD = CBD - - - 49
Th. the angle ACD = BCD - - - 62
Or the angle ACE = BCE - - - - -
Th. - - - - - - AE = EB - - - - 62

Problem 6.

298. *To erect a perpendicular to a given ſtrait line* AB, *at a given point in it* C.

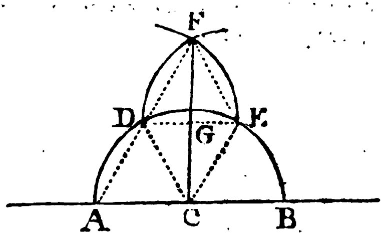


About the center C, at any diſtance, deſcribe a circle cutting the given line in D and E, about the centers D and E, with any radius greater than CD deſcribe circles cutting in F, and draw CF.

For join DF, FE.

Now CD is equal to CE - - - - 39
- - - DF is equal to FE - - - - con.
And CF is common - - - - - -
Th. CD, DF, FC = CE, EF, FC each to each. - con.
Th. the angle DCF = FCE - - - - 66
Th. CF is perpendicular to AB - - - 9

6. Otherwiſe.



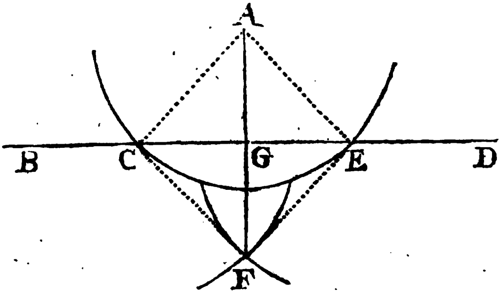
About the center C deſcribe a circle ADEB, on the center A cut the circle in D, on the center D deſcribe the arch FE, and on E cut that arch in F, all with the ſame radius, and draw CF.

For join AD, DF, FE, EC, CD, DE.

Now the triangles ADC, CDE, EDF are equilateral. con.
Th. the angles ACD, CDE, EDF are equal - - 66
Th. the angles DGF, DGC are equal - - 62
Th. DGF is a right angle - - - - 9
Ag. DE is parallel to AB - - - - 79
Th. the angle ACF is equal to DGF - - 83
Th. ACF is a right angle - - - - 54
Th. CF is perpendicular to AB - - - 9

Problem 7.

299. *From a given point* A, *to let fall a perpendicular upon an infinite ſtrait line* BD.

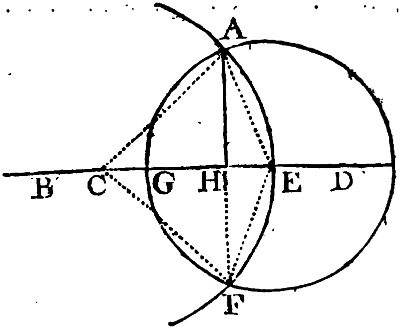


About the center A, deſcribe a circle at pleaſure cutting the given line in C and E, about the centers C and E, with the ſame radius deſcribe circles cutting one another in F, and draw AF cutting BD in G.

For join AC, CF, FE, EA.

Now AC, CF are equal to AE, EF - - con.
And the ſide AF is common - - - - -
Th. the angle CAF is equal to FAE - - 66
Or the angle CAG is equal to GAE - - -
Th. the angles CGA, AGE are equal - - 62
Th. AG is perpendicular to BD - - - 9

7. Otherwiſe.



In the given line BD aſſume the center C, with the radius CA deſcribe the circle AEF, cutting the line in E, about the center E, at the diſtance EA deſcribe the circle AGF, and draw AF cutting BD in H.

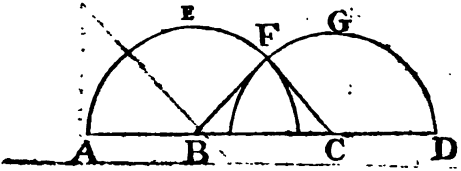
For join AC, CF, FE, EA.

Now CA is equal to CF - - - - - 30
- - - EA is equal to EF - - - - - 30
And the ſide CE is common - - - - - -
Th. the angle ACE = ECF - - - - 66
Th. the angle AHC = CHF - - - - - 62
Th. AH is at right angles to BD - - - 9

Problem

Problem 8.

300. *To make a triangle, whoſe ſides ſhall be equal to three given ſtrait lines* AB, BC, CD; *provided any two of them taken together be greater than the third.*



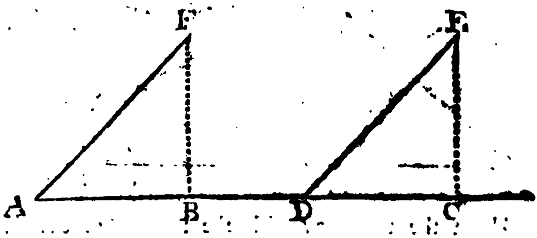
About the center B, with the radius BA, deſcribe the circle AEF; on the center C, with the radius CD, deſcribe the circle DGF, and draw BF, FC.

For BF is equal to BA - - - - - - 30
And CF is equal to CD - - - - - 30
Th. BFC is the triangle required.

Problem

Problem 9.

301. *At a given point* A, *in a given ſtrait line* AB, *to make an angle equal to a given angle* CDE.

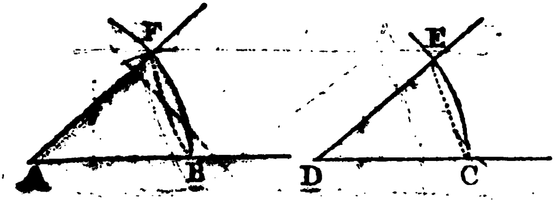


Make AB equal to DC, by problem 3, erect BF, CE perpendicular to AB, DC, each to each, by problem 6, make BF equal to CE, and draw AF.

For AB is equal to DC - - - - - con.
Alſo BF is equal to CE - - - - - con.
Th. AB, BF are equal to DC, CE each to each - -
And the contained angles ABF, DCE are equal - 54
Th. the angle BAF is equal to CDE - - 62

9. Otherwiſe.

At a given point A, in a given ſtrait line AB, to make an angle equal to a given angle CDE.



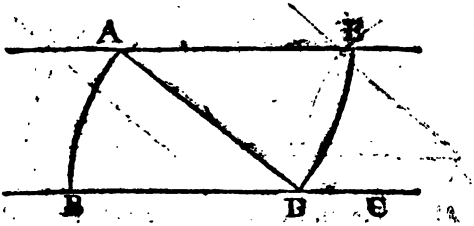
About the center D, with the radius DC, deſcribe the arch CE, about the center A, with the ſame radius DC deſcribe the arch BF, on the center B, with the diſtance CE cut BF in the point F, and draw AF.

For join BF, CE.

Now AB is equal to DC - - - - - con.
AF is equal to DE - - - - - con.
And BF is equal to CE - - - - - con.
Th. BA, AF, FB = CD, DE, EC each to each - -
Th. the angle A is equal to the angle D - - 66

Problem 10.

[illegible]. *Through a given point A, to draw a line parallel to a given strait line BC.*

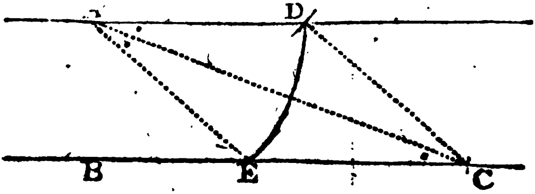


In the given line take the point D at pleasure, and join AD, make the angle DAE equal to the angle ADB, by problem 9, and the thing is done.

For the angle DAE = ADB . . . . . . constr.
And they are alternate angles . . . . . .
Th. AE is parallel to BC . . . . . [illegible]

16. Otherwiſe.

Through a given point A, *to draw a line parallel to a given ſtrait line* BC.



In the given line aſſume the point E, and about the center A, with the radius AE, deſcribe the arch ED, make EC equal to EA, on the center C, with the radius CE, cut the arch ED in D, and draw AD.

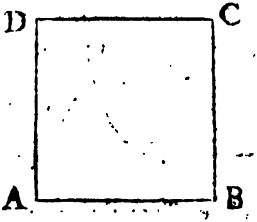
For join AE, AC, CD.

Now AD = AE, CD = CE - - - - - 19
And EC is equal to EA - - - - - con.
Th. DA, AE, EC, CD are equal - - - - [illegible]
And the ſide AC is common - - - - [illegible]
Th. CD, DA, AC = AE, EC, CA each to each [illegible]
Th. the angle CAD = ACE - - - - 66
Th. AD is parallel to BC - - - - 79

Problem

Problem 11.

130. *To deſcribe, or make a ſquare, upon a given ſtrait line* AB.



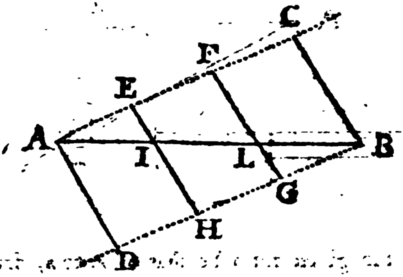
Draw AD at right angles to AB, by problem 6, make AD equal to AB, draw DC parallel to AB, and BC parallel to AD, by problem 10.

For ABCD is a parallelogram - - - - 22
Th. AB = DC, and AD = BC - - - 95
Th. AB, BC, CD, DA are equal - - - 48
And the angles are all right angles - - - 92
Th. ABCD is a ſquare - - - - 55

Problem

Problem 11.

To divide a given ſtrait line into any propoſed number of equal parts.



Draw AC at pleaſure, and BD parallel to AC, by problem 10, in AC, BD, take the propoſed number of equal parts of any length, namely, AE, EF, FC, BG, GH, HD, all equal to one another, and draw EH, FG cutting the given line in I and L.

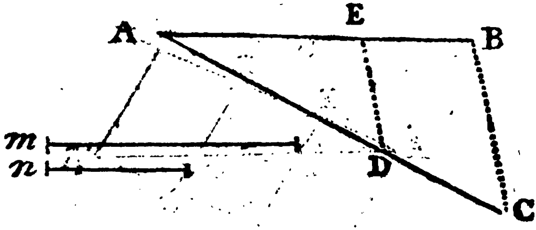
For join BC.

Now FC, GB are equal and parallel - - - con.
Th. FG is parallel to CB - - - - 90
By the ſame reaſon EH is parallel to CB - - -
But AE, EF, FC are equal - - - - con.
Th. AI, IL, LB are equal - - - - 101

Problem

Problem [illegible]

To divide a given ſtrait line AB *in a given ratio.*

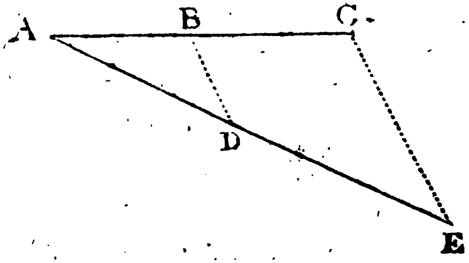


Let the given ratio be that of *m* to *n*, draw AC at pleaſure, and in it take AD equal to *m*, and DC equal to *n*, join BC, draw DE parallel to CB, and E will be the point required.

For AD : DC :: AE : EB . . . . . [illegible]
And AD = *m*, DC = *n* . . . . . [illegible]
Th. *m* : *n* :: AE : EB . . . . . [illegible]

Problem 14.

306. *To find a third proportional to two given ſtrait lines* AB, BC.



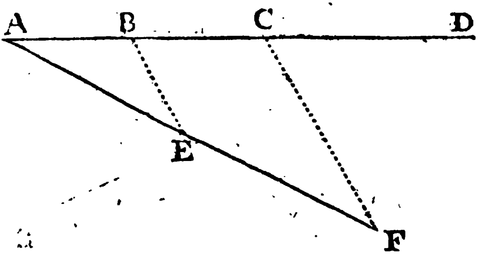
Let AB and BC make a ſtrait line, draw AE at pleaſure, and in it take AD equal to BC, by problem 3, join BD, draw CE parallel to BD, and DE will be the line required.

For AB : BC :: AD : DE - - - - - - 189
And AD is equal to BC - - - - - - con.
Th. AB : BC :: BC : DE - - - - - - 56

Problem

Problem 15.

307. *To find a fourth proportional to three given ſtrait lines* AB, BC, CD.

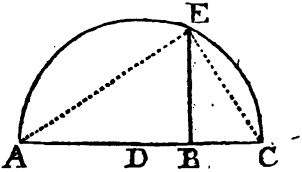


Let AB and BC make a ſtrait line, draw AF at pleaſure, and in AF take AE equal to CD, join BE, draw CF parallel to BE, and EF will be the line required.

For AB : BC :: AE : EF - - - - - 189
And AE is equal to CD - - - - - con.
Th. AB : BC :: CD : EF - - - - - 56

Problem 16.

308. *To find a mean proportional between two given ſtrait lines* AB, BC.



Let AB and BC make a ſtrait line, divide AC equally in D, by problem 5, about the center D, with the radius DA or DC deſcribe a ſemicircle AEC, and draw BE perpendicular to AC, by problem 6.

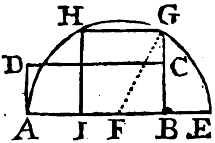
For join AE, EC.

Now AEC is a right angle - - - - 153
And EB is perpendicular to AC - - - con
Th. AB : BE :: BE : BC - - - - - 199
Th. BE is a mean proportional - - - 161

Problem

Problem 17.

309. *To make a ſquare equal to a given rectangle* ABCD.



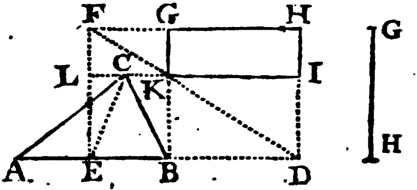
Produce AB to E, make BE equal to BC, biſect AB in F, about the center F deſcribe the circle AGE, continue BC to G, and upon BG deſcribe the ſquare BGHI.

For join FG.

Now $AB \times BE + FB^2 = FE^2$ - - - 111
And $BE = BC$, $FG = FE$ - - - con.
Th. $AB \times BC + FB^2 = FG^2$ - - - 56
But - - - - $BG^2 + FB^2 = FG^2$ - - - 100
Th. $BG^2 + FB^2 = AB \times BC + FB^2$ - - 48
Th. BG^2 - - - $= AB \times BC$ - - - 50
Or BGHI - - - = ABCD - - - - 61

Problem 18.

310. *To make a rectangle upon a given ſtrait line* GH, *that ſhall be equal to a given triangle* ABC.



Produce AB to D, make BD equal to the given line, biſect AB in E, erect EF at right angles to AB, draw BG, DH parallel to EF, and LCI parallel to AD, cutting BG in K, join DK, which continue to F, and draw FH parallel to BD; then GHIK is the rectangle required.

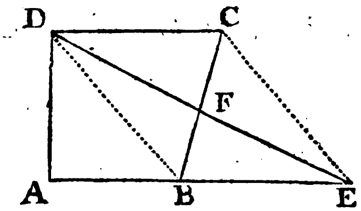
For join CE.

Now the triangle ACE = ECB	- - -	96
Th. the triangle ABC = 2ECB	- - -	49
But the rectangle EK = 2ECB	- - -	98
Th. the triangle ABC = EK	- - -	48
Ag. the rectangle GI = EK	- - -	99
Th. the rectangle GI = ABC	- - -	48
And the line - - GH = BD	- - -	91

Problem

Problem 19.

311. *To make a triangle equal to a given quadrangle* ABCD.



Produce AB to E, join DB, draw CE parallel to DB, and join DE.

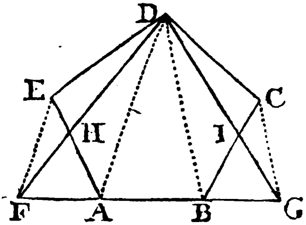
For the triangle BDE = BDC - - - 95
Th. the triangle BEF = FCD - - - 50
Th. the triangle AED = ABCD - - - 49

Remark.

By means of the preceding problem (18) *a rectangle may be made equal to any given ſtrait lined figure, firſt dividing the ſaid figure into triangles.*

Problem 20.

312. *To make a triangle equal to a given pentagon* ABCDE.



Produce the baſe AB both ways to F and G, join DA, DB, draw EF parallel to DA, CG parallel to DB, and join FD, DG.

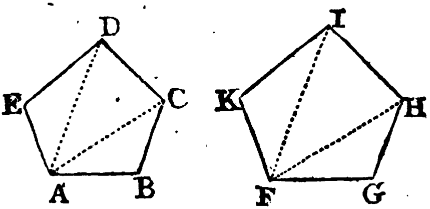
For the triangle ADF = ADE - - - 95
And the triangle BDG = BDC - - - 95
Th. the triangle AFH = HED - - - 50
And the triangle BGI = ICD - - - 50
Th. the triangle FGD = ABCDE - - - 49

Remark.

After the very ſame manner as this problem is ſolved, proceeding ſtep by ſtep, a triangle may be made equal to any given ſtrait lined figure.

Problem 21.

313. *Upon a given ſtrait line* FG, *to make a figure ſimilar to a given figure* ABCDE.

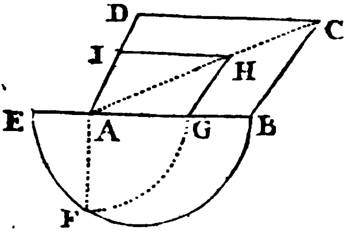


Join AC, AD, make the angle FGH equal to ABC, GFH equal to BAC, FHI equal to ACD, HFI equal to EAD, &c. by problem 9, and the thing is done.

For the angle FHG = ACB - - - - 87
And the angle FIH = ADC - - - - - 87
Th. the angle GHI = BCD - - - 49
And the angle HIK = CDE - - - - 49
Ag. AC : FH :: BC : GH - - - 193*
And AC : FH :: CD : HI - - - 193*
Th. BC : GH :: CD : HI - - - - 173
Th. FGHIK is ſimilar to ABCDE - - 165

Problem 22.

314. *To make a figure ſimilar to a given figure* ABCD, *and in the given ratio to it of m to n.*



Produce BA to E, find AE a fourth proportional to m, n, and BA, by problem 15, upon the diameter EB deſcribe the ſemicircle EFB, draw AF at right angles to EB, make AG equal to AF, and the figure AGHI, ſimilar to ABCD, by problem 21.

For $AB : AE :: AB^2 : AE \times AB$ - - 184

And AF^2, or $AG^2 = AE \times AB$ - - 216

Th. $AB : AE :: AB^2 : AG^2$ - - - 56

But $AB : AE :: m : n$ - - - - con.

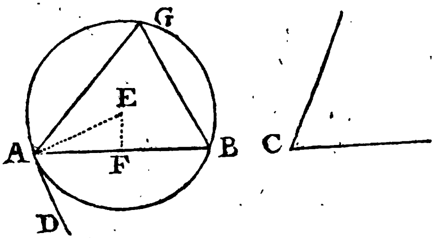
Th. $AB^2 : AG^2 :: m : n$ - - - - 173

Ag. $AB^2 : AG^2 :: ABCD : AGHI$ - - 212

Th. $m : n :: ABCD : AGHI$ - - - 173

Problem 23.

315. *Upon a given ſtrait line* AB, *to deſcribe the ſegment of a circle, that ſhall contain an angle equal to a given angle* C.



Make the angle BAD equal to the given angle C, draw AE at right angles to AD, biſect AB in F, draw FE perpendicular to AB, and about the center E, with the radius EA or EB deſcribe the ſegment AGB.

For join AG, GB.

Now AD is a tangent to the circle - - 138

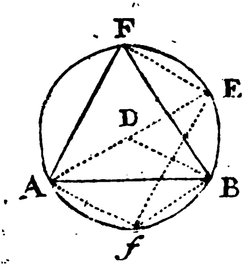
Th. the angle BAD = AGB - - - 155

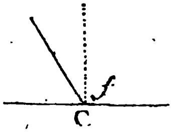
But the angle BAD = C - - - - con.

Th. the angle AGB = C - - - - 48

23. Otherwiſe.

Upon a given ſtrait line AB, *to deſcribe the ſegment of a circle, that ſhall contain an angle equal to a given angle* C.



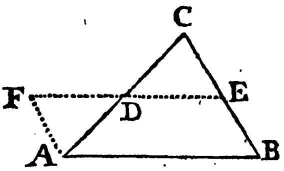


Make each of the angles ABD, BAD equal to the difference between the given angle C and a right angle; and about the center D, with the radius DA or DB, deſcribe the ſegment AFB.

For produce AD to E, and join AF, FB, FE.
Now AE is a diameter of the circle - - - con.
Th. the angle AFB differs from a right angle by BAE - - - - - - - - 154
Alſo, the angle C differs from a right angle by BAE. con.
Th. the angle AFB = C - - - - 50

Problem 24.

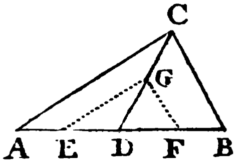
316. *The baſe of a triangle, the verticle angle, and the ratio of the ſides, are given to make the triangle.*



Make the angle ACB equal to the given angle, and CD to CE in the given ratio of *m* to *n*, join ED, and continue it to F, draw FA parallel to CB, and AB parallel to FE.

For the angle [illegible]DE = CAB - - - - 83
And the angle CED = CBA - - - - 83
Th. the triangles DCE, ACB are equiangular - 27
Th. DC : CE :: AC : CB - - - - 193
Alſo DC : CE :: *m* : *n* - - - - con.
Th. AC : CD :: *m* : *n* - - - - 173
And AB is equal to FE - - - - 91
Th. ABC is the triangle required.

317. *The angles at the baſe of a triangle, and a line joining the vertex and middle of the baſe, are given to make the triangle.*



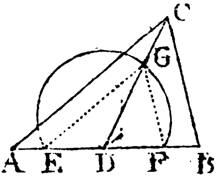
In the ſtrait line AB aſſume DE, DF equal to each other, make the angles FEG, EFG equal to the given angles at the baſe each to each, join DG and continue it to C, make DC equal to the given line, and draw CA, CB parallel to GE, GF.

For the triangle ACD is equiangular to EGD, and DCB is equiangular to DGF - - - - - 83
Th. CD : GD :: DA : DE - - - - *193
And CD : GD :: DB : DF - - - *193
Th. DA : DE :: DB : DF - - - - - 173
But DE is equal to DF - - - - - con.
Th. DA is equal to DB - - - - - 167
Th. ABC is the triangle required.

Problem

Problem 26.

318. *The verticle angle, a line joining the vertex and middle of the baſe, and the angle made by the ſaid line and baſe; are given to make the rectangle.*



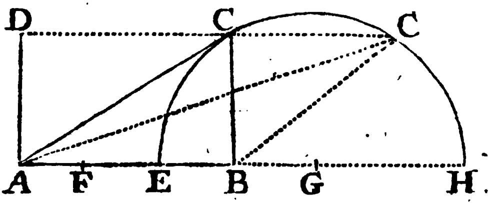
In the ſtrait line AB take DE, DF equal to each other, make the angle BDC equal to the given angle, and DC equal to the given line, upon EF deſcribe the ſegment of a circle that will contain the vertical angle, by problem 23, cutting DC in G, join EG, GF, and draw CA, CB parallel to GE, GF.

For the triangle ACD is equiangular to EGD, and
DCB is equiangular to DGF - - - 83
Th. the angle ACB = EGF - - - 49
Ag. CD : GD :: DA : DE - - - 193
And CD : GD :: DB : DF - - - - 193
Th. DA : DE :: DB : DF - - - - 173
But DE is equal to DF - - - - con.
Th. AD is equal to DB - - - - 167
Th. ABC is the triangle required.

Problem

Problem 27.

319. *The base of a triangle* AB, *the altitude* AD, *and the ratio of its ſides* AC, CB *are given as m to n*; *to make the triangle.*



Divide the baſe in the given ratio of the ſides at E, by problem 13, make EF equal to EB, produce AB to G, find EG a fourth proportional to AF, FE, AE, by problem 15, about the center G, with the radius GE deſcribe the circle ECH, draw DC parallel to AG, and join AC, CB.

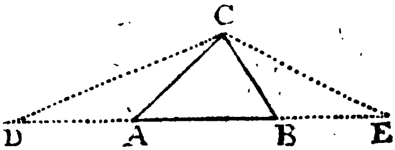
For *m* : *n* :: AE : EB - - - - - con.
And AC : CB :: AE : EB - - - - 219.
Th. *m* : *n* :: AC : CB - - - - 173.
Th. ABC is the triangle required.

Remark.

If the altitude AD *is leſs, equal to, or greater than* GE, *the radius of the circle*; *the problem accordingly has two anſwers, one anſwer only, or is impoſſible.*

Problem 28.

320. *The angles at the baſe of a triangle, and the ſum of its three ſides are given to make the triangle.*

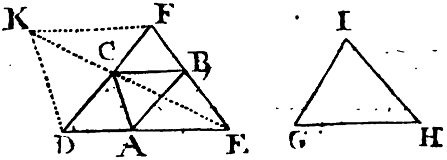


Let DE be the given ſum of the ſides: make the angles DEC, EDC equal to half the given angles at the baſe reſpectively, by problems 4 and 9; make the angle DCA equal to CDE, and the angle ECB equal to CED.

For the angle ABC = 2BEC - - - 86
And the angle BAC = 2ADC - - - 86
Th. the angles ABC, CAB are equal to the given angles - - - - - - - 51
Again, AC = AD, and BC = BE - - - 65
Th. AB + BC + CA = DE - - - - 49
Th. ABC is the triangle required.

Problem 29.

*321. *To inſcribe a triangle* ABC *in a given triangle* DEF, *and make it equiangular to another given triangle* GHI.



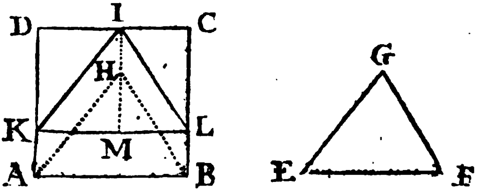
Make the angles FDK, DFK, equal to the acute angles G and H, each to each, join KE cutting DF in C, draw CA parallel to KD, CB parallel to KF, and join AB.

For the angle ACB = DKF - - - -	83
And the angle ECB = EKF - - - -	83
Th. the angle ACB = DKF - - - -	49
Ag. EK : EC :: DK : AC - - -	*193
And EK : EC :: KF : CB - - -	*193
Th. DK : AC :: KF : CB - - - -	173
Th. DK : KF :: AC : CB - - - -	175
Th. ABC is equiangular to DFK - - -	198
But DFK is equiangular to GHI - - -	con.
Th. ABC is equiangular to GHI - - -	48
And is inſcribed in DEF - - - -	44

Problem

Problem 30.

722. *To inſcribe a triangle in a ſquare* ABCD, *and make it equiangular to a given triangle* EFG.



Make the angles BAH, ABH equal to the two leaſt angles of the given triangle E and F, each to each, draw HI parallel to AD, IK parallel to HA, IL parallel to HB, and join KL.

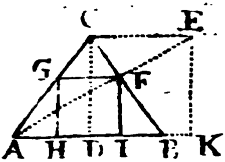
For produce IH to M.

Now KI is equal to AH - - - -	91
And IL is equal to HB - - - -	91
Alſo the angle KIM = AHM - - - -	83
And the angle MIL = MHB - - - -	83
Th. the angle KIL = AHB - - - -	49
Th. KLI is equiangular to ABH - - -	62
But ABH is equiangular to EFG - - -	con.
Th. KLI is equiangular to EFG - - -	48
And is inſcribed in the ſquare - - -	44

Problem

Problem 31.

323. *To inſcribe a ſquare in a given triangle* ABC.



Let fall the perpendicular CD, draw CE parallel to AB, make CE equal to CD, join AE cutting BC in F, draw FG parallel to BA, and GH, FI parallel to CD.

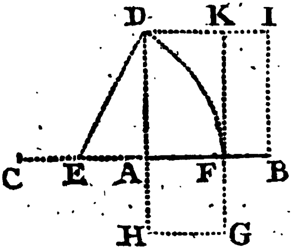
For produce AB to K, and draw EK parallel to CD.

Now AE : AF :: EC : FG - - - - *193
And AE : AF :: EK : FI - - - *193
Th. EC : FG :: EK : FI - - - - 173
But EC is equal to CD - - - - con.
And EK is equal to CD - - - - 91
Th. CD : FG :: CD : FI - - - - 56
Th. FG is equal to FI - - - - 167
But GH = FI, and HI = GF - - - 91
Th. FG, GH, HI, IF are equal - - - 48
And its angles are all right angles - - 92
Th. FGHI is a ſquare - - - - - 24

Problem

Problem 32.

324. To divide a given ſtrait line AB, *ſo that the rectangle contained by the whole line and one of the parts, ſhall be equal to the ſquare of the other part.*



Produce BA to C, draw AD at right angles to AB, make both AC and AD equal to AB, biſect AC in E, join ED, make EF equal to ED, and F is the point required.

For upon AB, AF, deſcribe the ſquares ABID, AFGH, by problem 11, and produce GF to K.

Now $AD^2 + AE^2 = ED^2$ - - - - 100

And $CF \times AF + AE^2 = EF^2$ - - - 114

But $ED^2 = EF^2$ - - - - - 105

Th. $AD^2 + AE^2 = CF \times AF + AE^2$ - - 48

Th. $AD^2 = CF \times AF$ - - - - 50

But $DH = CF$, and $HG = AF$ - - con.

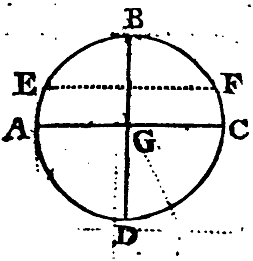
Th, $AD^2 = DH \times HG$ - - - - 56

Th. $KB = AG$, or $BI \times BF = AF^2$ - - 50

Th. $AB \times BF = AF^2$ - - - - 56

Problem 35.

325. *To find the center of a given circle* ABCD.

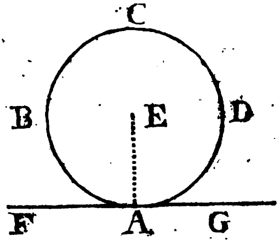


Draw EF at pleaſure cutting the circle in E and F, biſect EF at right angles, with the ſtrait line BD, by problem 5; again, biſect BD at right angles, by the line AC in G, and the point G will be the center of the circle.

For the center is in the ſtrait line BD - - 128
It is alſo in the ſtrait line AC - - - 128
Then G muſt be the center of the circle.

Problem 34.

326. *From a given point* A, *to draw a tangent to a given circle* BCD.



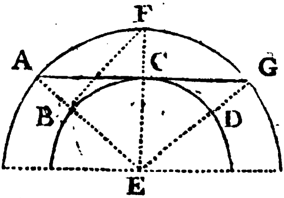
Caſe 1. *When the given point is in the circumference.*

Find E the center of the circle, by problem 33, join EA, and thro' the point A, draw FG at right angles to EA, by problem 6.

Then FG is a tangent to the circle. - - 138.

Problem 34.

Caſe 2. *When the given point* A, *is without the circumference* BCD.

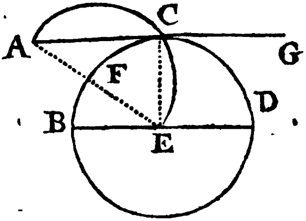


Find E the center of the circle, join AE, about the center E, at the diſtance EA deſcribe the circle AFG, draw BF at right angles to AE, join EF cutting the circumference in C, and draw AC.

For AE is equal to EF - - - - 30
- - - EC is equal to EB - - - - 30
Th. AE, EC = FE, EB each to each - -
And the contained angle AEC is common -
Th. the angle ACE = EBF - - - - 62
Th. ACE is a right angle - - - - 54
Th. AC is a tangent to the circle - - 138

34. Otherwiſe.

From a given point A, *to draw a tangent to the circle* BCD.



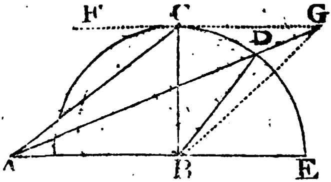
Find E the center of the circle, join AE, biſect AE in F, about the center F, with the diſtance FA, deſcribe the ſemicircle ACE, and draw AC.

For join CE.

Now AEC is a ſemicircle - - - -	con.
And the angle ACE is contained in it - -	con.
Th. ACE is a right angle - - - -	153
Th. AC is a tangent to the circle - -	138

Problem 35.

327. *To make the greateſt triangle, that ſhall have two given lines* AB, BC *for two of its ſides.*



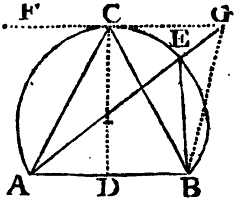
Place the given ſides AB, BC at right angles to each other, by problem 6, and join AC.

For let ABD be another triangle, having the ſide BD equal to the ſide BC, about the center B, with the radius BC deſcribe a circle CDE, draw FG a tangent to the circle at the point C, by problem 34, produce AD to G, and join BG.

Now BCG is a right angle - - - - - 240
Th. the angles ABC, BCG are equal - - - - 54
Th. AB, FG are parallels - - - - - 179
Th. the triangles ABG, ABC are equal - - - 195
But the triangle ABG is greater than ABD - - 152
Th. the triangle ABC is greater than ABD - - 56

Problem 36.

328. *To make the greateſt triangle upon a given baſe* AB, *with a given vertical angle* ACB.



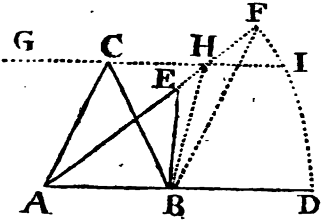
Upon the baſe AB, deſcribe the ſegment of a circle ABC that will contain the given angle, by problem 23, biſect AB in D, draw DC at right angles to AB, and join AC, CB.

For let ABE be another triangle in the ſegment, draw FG touching the circle in C, by problem 34, produce AE to G, and join BG.

Now DC paſſes thro' the center of the circle - 128
Th. DCG is a right angle - - - - 140
Th. AB, FG are parallels - - - - 79
Th. the triangles ABG, ABC are equal - - 95
But the triangle ABG is greater than ABE - - 52
Th. the triangle ABC is greater than ABE - - 56

Problem 37.

339. *To make the greateſt triangle upon a given baſe* AB, *with a given line* AD, *for the ſum of its ſides* AC, CB.



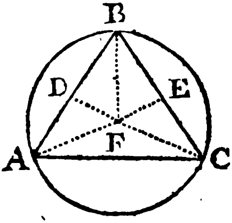
By problem 8, make an iſoſceles triangle ABC, whoſe ſides AC, BC are each equal to half AD, and the thing is done.

For let ABE be ſome other triangle, having AE + EB equal to AC + CB, produce AE to F, make EF equal to EB, or AF equal to AD, draw GCHI parallel to AB, cutting AF in H, and join BF, BH.

Now 'tis plain the angles ACG, BCH are equal - cor.
Th. AH + HB is greater than AC + CB - - 78
Th. AH + HB is greater than AF, or AH + HF - 56
Th. - - - HB is greater than - - - - - - - HF - 50
Th. the angle EFB is greater than FBH - - 72
But the angles EFB, FBE are equal - - 63
Th. the angle FBE is greater than FBH - - 56
Th. the point E is between A and H - - -
Th. the triangle ABH is greater than ABE - - 52
riangles ABH, ABC are equal - - 55
riangle ABC is greater than ABE - - 56

Problem 38.

330. *To circumſcribe a circle about a given triangle* ABC.



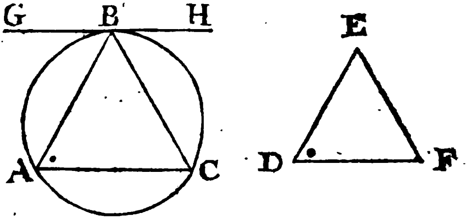
Biſect any two ſides AB, BC in D and E, draw DF, EF at right angles to AB, BC, and about the center F, with the radius FA deſcribe the circle.

For join AF, FB, FC.

Now AD is equal to DB - - - - con.
The ſide DF is common - - - - -
And the angle ADF = FDB - - - con.
Th. AF is equal to FB - - - - 62
By the ſame reaſon FB = FC - - - - -
Th. FA, FB, FC, are equal - - - - 48
Th. the circle paſſes through the points A, B, C 30
Th. the circle circumſcribes the triangle - - 41

Problem 39.

331. *In a given circle* ABC, *to inſcribe a triangle that ſhall be equiangular to a given triangle* DEF.



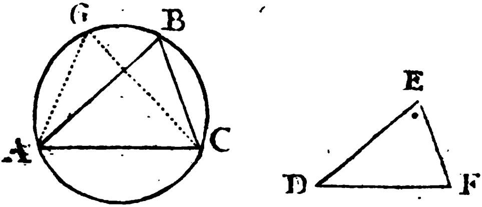
Draw GH a tangent to the circle at the point B, by problem 34, make the angle HBC equal to EDF, the angle GBA equal to DFE, and join AC.

For the angle HBC = BAC - - - 155
And the angle GBA = BCA - - - 155
Th. the angle BAC = EDF - - - - 48
And the angle BCA = EFD - - - 48
Th. the angle ABC = DEF - - - - 87
Th. ABC is equiangular to DEF - - - 87
And is inſcribed in the circle - - - 40

Problem

39. Otherwiſe.

In a given circle ABC, *to inſcribe a triangle that ſhall be equiangular to a given triangle* DEF.



Aſſume the point G in the circumference, and join AG, make the angle AGC equal to E, draw AC, make the angle CAB equal to D, and join BC.

For the angle AGC is equal to ABC - - 143

And the angle AGC is equal to E - - - con.

Th. the angle ABC is equal to E - - - 48

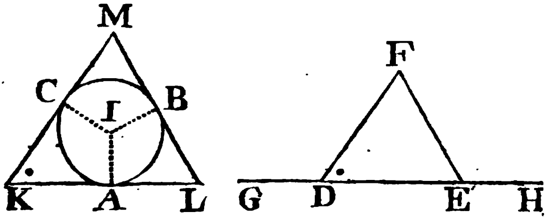
Alſo the angle CAB is equal to D - - con.

Th. the angle ACB is equal to F - - - 87

Th. the triangle ABC is equiangular to DEF - 27

Problem 40.

332. *To deſcribe a triangle about a given circle* ABC, *and make it equiangular to a given triangle* DEF.

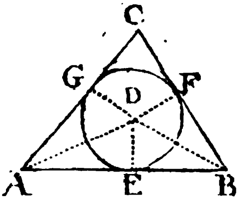


Produce the baſe DE both ways to G and H, find I the center of the circle, join IA, make the angles AIC, AIB equal to GDF, FEH each to each, and thro' the points A, B, C draw KL, LM, KM tangents to the circle.

For the angles of a quadrangle altogether, are equal to four right angles - - - - - - 89
And KAI, ICK are right angles - - cor.
Th. the angle AIC + CKA = two right angles 56
Alſo, the angle GDF + FDE = two right angles 67
Th. the angle AIC + CKA = GDF + FDE - 48
Th. the angle - - - - CKA = - - - - FDE - 50
By the ſame reaſon BLA = FED - - -
Th. the angle - - - - KML = DFE - - 87
Th. the triangle KLM is equiangular to DEF - - 27
And is deſcribed about the circle - - 42

Problem 41.

333. *To inſcribe a circle in a given triangle* ABC.

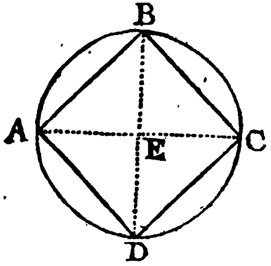


Biſect any two of the angles ABC, CAB, by the lines BD, AD, draw DE perpendicular to AB, and about the center D, with the radius DE deſcribe the circle.

For upon BC, CA let fall the perpendiculars DF,
DG - - - - - - - p. 7
Now the angle EBD = DBF - - - con.
- - - the angle DEB = BFD - - - - 54
And the ſide BD is common - - - - -
Th. DE is equal to DF - - - - 77
By the ſame reaſon DE = DG - - - -
Th. DE, DF, DG are equal - - - - 48
And AB, BC, CA are tangents to the circle - 138
Th. the circle EFG is inſcribed in the triangle ABC. 43

Problem 42.

334. *To inſcribe a ſquare in a given circle* ABCD.

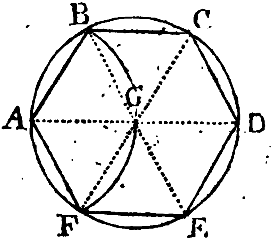


Draw the diameters AC, BD at right angles to each other, and join AB, BC, CD, DA.

For EA, EB, EC, ED are equal - - - 30
And the angles AEB, BEC, CED, DEA are equal 54
Th. AB, BC, CD, DA are equal - - - 62
Ag. ABC, BCD, CDA, DAB are right angles - 153
Th. the oppoſite ſides are parallel - - 81
Th. ABCD is a ſquare - - - - 24

Problem 43.

335. *To inſcribe a hexagon in a given circle* ABCD, &c.



Find G the center of the circle, draw the diameter AD, about the center A, with the radius AG, deſcribe a circle BGF, join BG, FG, produce them to E and C, and draw AB, BC, CD, DE, EF, FA.

For AB, AG, GB are equal - - - con.

Th. the triangle ABG is equiangular - - 64

Th. each angle is $\frac{1}{3}$ of two right angles - - 86

Th. - - - AGF is $\frac{1}{3}$ of two right angles, by the ſame reaſon - - - - - - - -

But the angle FGB + BGC = two right angles - 67

Th. - - - BGC is $\frac{1}{3}$ of two right angles - - 50

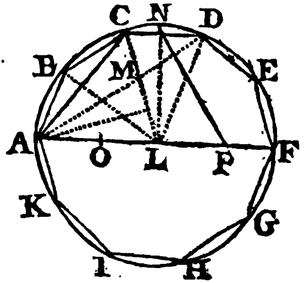
Th. BC is equal to AB - - - - - 62

By the ſame reaſon CD, DE, &c. are equal to AB - -

Th. AB, BC, CD, DE, EF, FA are equal - - 48

Problem 44.

336. *To inſcribe a regular pentagon, or decagon, in a given circle.*



Inveſtigation.

Let ABCD, &c. be the inſcribed decagon, join AC, AD, find L the center of the circle, and draw BL, LC, LD.

Now the arches AB, BC, CD, &c. are equal - - 151
Th. the angles ALB, BLC, CLD are equal - - 150
And the angles ADC, DAF are equal - - 150
But the angle ADC is equal to ALB, or BLC - 142
Th. the angles ADC, DAF, CLD are equal - - 48
And CD, AF are parallels - - - - 79
Th. the angles ALM, MCD are equal - - 82
Th. the triangles ALM, LCD, DCM are equiangular. 87
Th. LC : CD :: DC : CM - - - - 193
And AM, ML are equal to LD, DC each to each - 77
Th. LC : LM :: LM : CM - - - - 56
Th. LC × CM = LM² - - - - 186

Ag. AM is equal to AL - - - - 43
Th. $AC^2 - AL^2 = LC \times CM$ - - - 121
Th. $AC^2 = AL^2 + LC \times CM$ - - - 49
Th. $AC^2 = AL^2 + LM^2$ - - - - 56

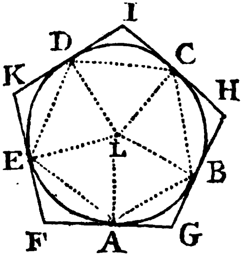
Whence by problem 3~~1~~, we have the following /2 conſtruction.

Draw the diameter AP, and the radius LN at right angles to it, biſect AL in O, make OP equal to ON, and join NP.

Then $LP = (ML =)$ CD is the ſide of the decagon, and $NP = AC$, is the ſide of the pentagon, which muſt be applied round the circumference of the circle.

Problem 45.

337. *About a given circle to deſcribe a regular figure, having the ſame number of ſides with a regular figure inſcribed in that circle.*



Let A, B, C, D, E be the angular points of the inſcribed figure, and at the points A, B, C, D, E, by problem 34, draw tangents to the circle, meeting one another at F, G, H, I, K.

For let L be the center of the circle.

Now the angles LEF, FAL, LAG, GBL are equal 54
And the angles LEA, LAE, LAB, LBA are equal 66
Th. the angles FEA, FAE, GAB, GBA are equal 50
Th. the angles F, G are equal, and the ſides EF, FA, AG, GB are equal - - - - - 77
By the ſame reaſon, the angles G, H, are equal, and AG, GB, BH, HC are equal - - - - -
Th. FG, GH, HI, &c. are equal - - - 49
Th. the figure FGHIK is regular - - - 26
And is deſcribed about the circle - - - 42

END of the GEOMETRY.

BOOK VII.

Of MEASURING.

338. *LINES, lengths, or dimenſions, are eſtimated or meaſured by inches, feet, yards, &c.*

339. *A ſquare, whoſe ſide is in length one inch, one foot, one yard, &c. is called a ſquare inch, ſquare foot, ſquare yard, &c.*

340. *A cube, whoſe ſide is in length one inch, one foot, one yard, &c. is called a cubic inch, cubic foot, cubic yard, &c.*

341. *Surfaces are eſtimated or meaſured by the number of ſquare inches, feet, yards, &c. which they contain.*

342. *Solids are eſtimated or meaſured by the number of cubic inches, feet, yards, &c. which they contain.*

Problem 1.

343. *To multiply feet, inches and parts, by feet, inches and parts, which method is termed croſs multiplication.*

RULE.

Set the feet in the multiplier, under the leaſt denomination in the multiplicand, and the reſt in order; multiply as in common arithmetick, divide each product by 12 (as you go on) place the firſt remainder under the multiplying figure, and the reſt in order, adding the ſeveral quotients to the next ariſing products; and having thus finiſhed the multiplication, the ſum of all will be the product required.

EXAMPLE.

What is the product of 7 feet, 6 inches, 9 parts, by 6 feet, 5 inches, 3 parts?

f. ′ ″
Now the multiplicand is 7 : 6 : 9
- - - the multiplier is 6 : 5 : 3 duly placed

1 : 10 : 8 : 3
3 : 1 : 9 : 9 :
45 : 4 : 6 :

The product is - - 48 : 8 : 2 : 5 : 3

EXAMPLE 2.

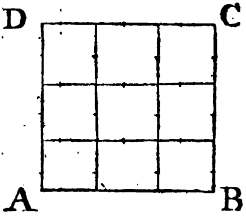
f. i. ″ f. i. ″
Multiply - - 9 : 7 : 3 - - by 7 : 9 : 3
The multiplier is - - 7 : 9 : 3 properly placed

2 : 4 : 9 : 9
7 : 3 : 5 : 3 :
67 : 2 : 9 :

The product is 74 : 8 : 7 : 0 : 9

Problem 2.

344. *To find the area, or content of a ſquare* ABCD.



RULE.

Multiply the length of the ſide by itſelf, and the product will expreſs the area.

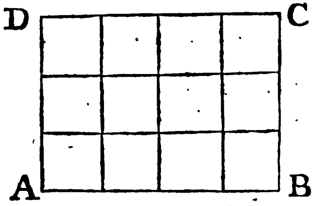
For if two adjacent ſides of the ſquare AB, AD are divided into the ſame number of equal parts, and from the points of diviſion lines be drawn parallel to the other ſides AD DC; it is evident that the given ſquare will be divided into leſſer ſquares, that the number of little ſquares in a row is equal to the number of parts (or meaſures) in the ſide, that the number of rows is likewiſe equal to the number of parts in the ſide, and conſequently, the ſquare of that number will expreſs the area.

EXAMPLE.

Let the ſide AB be three meaſures of any kind; then $AB \times AB = 3 \times 3 = 9$ the area.

Problem 3.

345. *To find the area of a rectangle or oblong* ABCD.



RULE.

Multiply the length by the breadth, namely, the baſe by the perpendicular; and the product will expreſs the area.

For if the baſe AB, and perpendicular AD, are divided into parts of equal length, and parallels be drawn to the other ſides AD, DC, from the points of diviſion, it is evident that the rectangle is divided into little ſquares, that the number in a row upon the baſe, is equal to the number of parts or meaſures in it, the number of rows is equal to the number of parts in the perpendicular, and conſequently the product of thoſe numbers will expreſs the area.

EXAMPLE.

Let the baſe AB be 4, and the perpendicular BC be 3. Then $AB \times BC = 4 \times 3 = 12$, the area.

EXAMPLE 2.

What is the area of a rectangular floor, whoſe length is 33 feet 9 inches, and breadth is 22 feet 6 inches?

1. By croſs multiplication.

f. i.
The length or baſe is - - 33 : 9
The breadth or perpendicular is - - : 22 : 6

16 : 10 : 6
742 : 6

The area is - - - 759 : 4 : 6

2. By decimals.

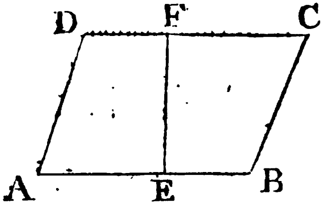
The length or baſe is - - 33,75
The breadth or perpendicular is 22,5 ×

16875
6750
6750

The product is - - - 759,375. the area as before.

Problem 4.

346. *To find the area of a parallelogram* ABCD.



RULE.

Multiply the length by the breadth, and the product will give the area.

Becauſe a parallelogram is equal to a rectangle of the ſame baſe and altitude - - - - 94

EXAMPLE.

Let the length or baſe AB be 242 yards, and the breadth or perpendicular EF be 160 yards; to find the area.

Length AB - - - 242
Breadth EF - - - 160

14520
242

The product is - - 38720 the area required.

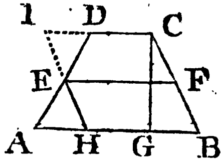
REMARK.

If the area of a field in yards is divided by 4840 (the number of ſquare yards in one acre) the quotient will give the number of acres in that field. In this example there are eight acres.

A

A Theorem.

347. *Every quadrangle having two parallel ſides, is equal to a rectangle contained by half their ſum, and a perpendicular between them.*



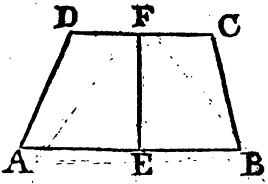
Let ABCD be a quadrangle having the ſide DC parallel to the baſe AB, biſect AD in E, draw EF parallel to AB or DC, and let fall the perpendicular CG; then EF = half $\overline{AB+DC}$, and ABCD = EF×CG.

For thro' the point E draw HI parallel to BC, and produce CD to I.

Now the angles AHE, EAH are equal to DIE, EDI, 82
And the correſponding ſides AE, ED are equal, con.
Th. AH = ID, and the triangle AHE = EID - 77
Th. ABCD = HBCI - - - - - - 49
But HB×CG = HBCI - - - - - 94
Th. ABCD = HB×CG - - - - - 48
Again, EF, HB, IC are equal - - - 91
Th. 2EF = HB + ID + DC - - - - 49
Or 2EF = HB + AH + DC = AB + DC - - 48
Th. EF = half $\overline{AB+DC}$ - - - - 51
And ABCD = EF × CG - - - - 56

Problem 5.

348. *To find the area of a quadrangle* ABCD, *having two parallel ſides* AB, DC.



RULE.

Multiply half the ſum of the parallel ſides by the perpendicular between them; and the product will give the area.

EXAMPLE.

Let the parallel ſides AB, DC, and the perpendicular between them EF be 955, 637, and 630 links of Gunter's chain reſpectively; to find the area of the figure.

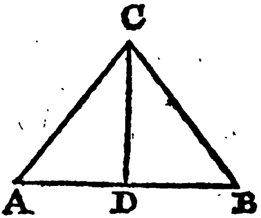
Now AB	- -	=	955
- - - DC	- -	=	637
Their ſum	- -	=	1592
The half ſum	- -	=	796
The perpendicular EF		=	630 ×
			23880
			4776
The product	- -	=	501480. the area in links.

REMARK.

If the area of a field in links is divided by 100000 (*the number of ſquare links in one acre*) *the quotient will expreſs the number of acres in that field—Thus in the example above, the field contains five acres.*

Problem 6.

349. *To find the area of a triangle* ABC.



RULES.

1. *Multiply the baſe by half the altitude, or the altitude by half the baſe, and the product gives the area.* Or

2. *Half the product of the baſe and altitude will give the area.*

Becauſe a triangle is equal to half a parallelogram of the ſame baſe and altitude - - - - 98

EXAMPLE.

Let the baſe AB be 97, and the altitude or perpendicular CD be 68, to find the area.

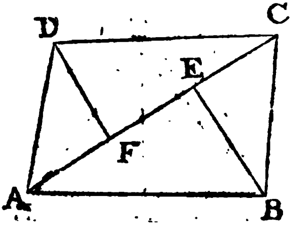
Now AB = 97 - - - AB = 97 } ſecondly.
Half CD = 34 - - - CD = 68 }

388   776
291   582

The area = 3298 - - - = half 6596 = 3298.

Problem 7.

350. *To find the area of any quadrangle* ABCD.



RULE.

Meaſure a diagonal line AC, *and the perpendiculars* BE, DF *falling upon it from the oppoſite angles; multiply the ſum of theſe perpendiculars by half the diagonal, and the product will give the area; or multiply the ſum of the perpendiculars by the diagonal, and half the product will give the area.*

This rule ariſes from the preceding, and is only determining two triangles at one operation.

EXAMPLE.

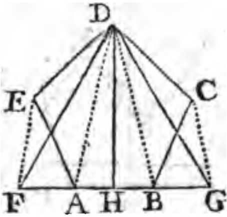
Let AC, BE, DF be 18, 8, and 7 of Gunter's chain reſpectively, to find the area.

Now 8 + 7 = 15 the ſum of the perpendiculars.

Th. 15 × 9 = 135, the area in chains, which divided by 10 (the number of ſquare chains in one acre) gives 13,5 acres; namely, thirteen acres and a half.

Problem 8.

351. *To find the area of any ſtrait lined figure.*



RULES.

1. *Divide the figure into triangles, find the area of each triangle, by problem* 6, *and their ſum will be the content.* Or

2. *Make a triangle equal to the given figure, by article* 312, *and find the area of this equal triangle.*

EXAMPLE.

Let ABCDE be the given figure, and FGD be a triangle made equal to it, whoſe baſe FG meaſures 1244 links, and perpendicular DH meaſures 1120 links.

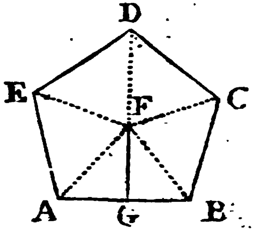
Then FG - - - - = 1240
Half DH - - - - = 560

74400
6200

The area of FGD, or ABCDE - = 694400 links.
Which is - - - - 6,94400 acres.

Problem 9.

352. *To find the area of any regular polygon.*



RULE.

Let fall a perpendicular from the center of the figure to one of its ſides; then multiply together the perpendicular, the ſide of the figure, and the number of its ſides; and half the product will expreſs the area.

For if lines be drawn from every angular point to the center of the figure, the polygon will be divided into the ſame number of equal triangles, as it has ſides.

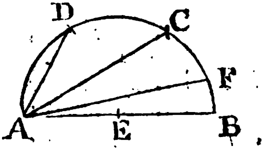
EXAMPLE.

What is the area of a regular pentagon ABCDE, whoſe ſide AB is 250, and perpendicular FG is 172?

Now 172×250×5 = 215000 the product of the three given quantities, and its half is 107500, the content required.

Problem 10.

353. *To find the circumference of a circle whoſe diameter is 2.*



Inveſtigation.

In the ſemicircle ABCD, apply the line AD equal to the radius AE, and draw AC biſecting the angle BAD.

Now AD is the ſide of a regular hexagon (335) and the arches BC, CD are equal (148) therefore each is $\frac{1}{6}$ of the whole circumference, and the line AC is given by 220. By the ſame theorem, a ſeries of biſecting lines may be found approaching to the diameter AB, and every intercepted arch (BC) will be a known part of the circumference; and having thus determined an arch BF ſufficiently minute for our purpoſe, by joining BF, the triangle AFB is right angled at F (153) and the line BF is given (100.) Laſtly, BF being the ſide of a regular inſcribed figure, whoſe number of ſides is given, the circuit of that polygon is given, and conſequently the circumference of the circle (being greater than the circuit of any inſcribed polygon) is nearly found, but to determine it exactly is impoſſible.

Opera-

Operation.

Nº.	AC²	AC	Ar. BF:
1	3,0000000000	1,7320508075	$\frac{1}{6}$
2	3,7320508075	1,9318516525	$\frac{1}{12}$
3	3,9318516525	1,9828897227	$\frac{1}{24}$
4	3,9828897227	1,9957178465	$\frac{1}{48}$
5	3,9957178465	1,9989291743	$\frac{1}{96}$
6	3,9989291743	1,9997322757	$\frac{1}{192}$
7	3,9997322757	1,9999330678	$\frac{1}{384}$
8	3,9999330678	$BF^2 = 0,0000669322$	$\frac{1}{768}$

Therefore BF = 0,00818121, and 0,00818121 × 768 = 6,28317, the circuit of an inſcribed polygon, having 768 ſides.

Again, ſince the inſcribed polygon of 768 ſides is determined, the circuit of a circumſcribing polygon ſimilar to it is eaſily found to be 6,28322, (the circuits and perpendiculars from the center to their ſides being proportional) and conſequently the circumference of the circle is nearly 6,2832, which is a mean between them.

Corollaries.

Corollaries.

354. *If the diameter of a circle is* 1, *the circumference is* 3,1416.

355. *If the diameter of a circle is multiplied by* 3, 1416 *the product will give the circumference.*

356. *If the radius of a circle is multiplied by* 6,2832, *the product will give the circumference.*

357. *As* 7 : 22 :: *diameter* : *circumference.*

358. *As* 7 : 44 :: *radius* : *circumference.*

359. *If the circumference of a circle is divided by* 3, 1416, *the quotient will give the diameter.*

360. *If the circumference of a circle is divided by* 6. 2832, *the quotient will give the radius, or ſemidiameter.*

Becauſe the circumferences of circles are proportional to their diameters, or ſemidiameters - - 225

Remark.

Since the circumference of the circle is only determined nearly, and not accurately; ſo the corollaries above are nearly true only, but not exactly ſo.

Problem

Problem 18.

361. *To find the area of a circle, whoſe diameter and circumference are given.*

RULES.

1. *Multiply half the circumference, by half the diameter, and the product will expreſs the area.* Or,

2. *Multiply the circumference by the diameter, and a fourth part of the product will expreſs the area.*

Becauſe a circle is equal to a triangle, whoſe baſe is equal to the circumference, and altitude is equal to the ſemidiameter. - - - - - - 227

EXAMPLE.

What is the area of a circle, whoſe diameter is 2, and circumference is - - 6,2832?

Now half the circumf. = 3,1416

- - - half the diameter = - - - 1

Th. the area - - - = 3,1416 by rule firſt.

EXAMPLE 2.

What is the area of a circle, whoſe diameter is 1, and circumference is - - 3,1416?

Multiply by the diameter - - 1

The product is - - = 3,1416

Th. the area - - = 0,7854 by rule ſecond.

Problem 12.

362. *The diameter, or ſemidiameter of a circle being given, to find the area of that circle.*

RULES.

1. *Multiply the ſquare of the diameter by* 0,7854, *and the product will give the area.* Or

2. *Multiply the ſquare of the ſemidiameter by* 3,1416, *and the product will give the area.*

Becauſe 0,7854 and 3,1416 are the areas of circles, whoſe diameters are 1 and 2 (361) and the areas of circles are proportional to the ſquares of their diameters, or ſemidiameters - - - - 225

Again, 1 : 0,7854 :: 14 : 11, nearly,
And 1 : 3,1416 :: 7 : 22, nearly.

Hence the following RULES.

3. *As* 14 *to* 11, *ſo is the ſquare of the diameter to the area of the circle* - - - - - 173

4. *As* 7 *to* 22, *ſo is the ſquare of the ſemidiameter to the area of the circle* - - - - - 173

REMARK.

The area of a circle cannot be found exactly, becauſe the diameter and circumference, are not (both of them) to be accurately expreſſed by numbers - - - 353

EXAMPLE.

What is the area of a circle whoſe diameter is 12?

Now 12 × 12 = 144, the ſquare of the diameter, and 0,7854 × 144 = 113,0976, the area required, by rule the firſt.

Secondly, 6 × 6 = 36, the ſquare of the ſemidiameter; and 3,1416×36 = 113,0976, the area required, by rule the ſecond.

Thirdly, 14 : 11 :: 12 × 12; area of the circle, which therefore $= \frac{12\times12\times11}{14} = \frac{1484}{14} =$ 113,1, by rule the third.

Laſtly, 7 : 22 :: 6 × 6 : area of the circle, which conſequently $= \frac{6\times6\times22}{7} = \frac{792}{7} =$ 113,1, by rule the fourth.

Problem

Problem 13.

363. *The circumference of a circle being given, to find the area.*

RULES.

1. *Find the ſemidiameter by the* 360, *and then find the area by the* 361. Or

2. *Multiply the ſquare of the circumference by* 0,079577, *and the product will give the area.*

For the ſquares of the circumferences, are as the ſquares of the diameters (207) therefore the areas are as the ſquares of the circumferences (178) and 0,079577 is the area of a circle whoſe circumference is 1.

EXAMPLE.

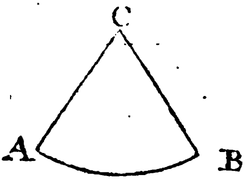
What is the area of a circle whoſe circumference is 24?

Now the ſemidiameter is 3,8197, and 3,8197 × 12 = 45,8364 the area required, by rule the firſt. Or

Secondly, the ſquare of the circumference 24 × 24 = 576, and 0,079577 × 576 produces 45,836352 for the area required.

Problem 14.

364. *To find the area of a ſector of a circle* ABC.



RULE.

Multiply the length of the arch by the radius of the circle, and half the product will give the area. Or multiply either of them by half the other, and the product will expreſs the area.

For a ſector of a circle is equal to a triangle, whoſe baſe is equal to the length of the arch, and altitude is equal to the radius of the ſector - - - 227

EXAMPLE.

Let the radius CA be 55, and the length of the arch AB be 59.

Now the arch AB - - = 59
The radius CA - - = 55

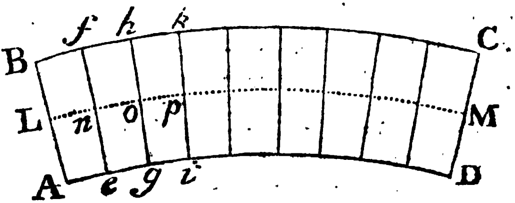
295
295

- - - the product - - = 3245
Half the product - - = 1622,5 the area of ABC.

Problem

Problem 15.

365. *To find the area of a ſegment of a ſector* ABCD, *or the front of an arch built with ſtones of equal length.*



RULE.

Multiply half the ſum of the bounding arches AD, BC; *by their diſtance* AB, *and the product will give the area.*

For let the ſegment be divided into equal parts indefinitely ſmall, by ſtrait lines e f, g h, i k, &c. drawn from the common center of the arches AD, BC; and about the ſaid center deſcribe the arch LM biſecting AB or DC, and cutting e f in n.

Now the parts A f, e h, g k, &c. repreſenting the fronts of arch ſtones indefinitely thin, they may be taken for quadrangles, having their upper and lower ſides parallels, and being all equal to one another, each is equal to L n × AB (347); therefore the whole ſegment ABCD = LM × AB (106) and L n being half the ſum of B f and A e, LM muſt be half the ſum of BC and AD; which gives the rule.

EXAMPLE.

EXAMPLE.

What is the area of the front of an arch, built with ſtones of 4 feet long, whoſe upper and lower bounding arches are in length 91 and 78 ½ feet reſpectively?

Now the upper curve - - - = 91
- - - - - lower curve - - = 78,5

The ſum - - - = 169,5
The half ſum - - = 84,75
Multiply by - - - 4

The area required - - 339 ſquare feet

EXAMPLE 2.

What is the area contained between two concentric ſemicircles, whoſe diameters are 24 and 16?

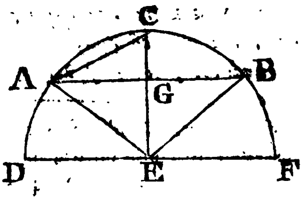
Now the greater ſemicircle - - = 37,7 } by 357
- - - the leſſer ſemicircle - - = 25,1 }

Their ſum - - - - = 62,8

The half ſum - - - = 31,4
Multiply by - - - - 4 the diſtance between the circles, and the product 125,6 is the area required.

Problem 16.

366. *To find the area of a ſegment of a circle* ABC, *whoſe center is* E.



RULES.

1. *Find the area of the triangle* ABE, *and of the ſector* ACBE, *by articles* 349, 364; *and their difference is the area of the ſegment* ABC. *Or*

2. *To ſix times the baſe, add eight times the chord of half the arch, multiply the ſum by the altitude, divide the product by* 15, *and the quotient will nearly give the area.*

EXAMPLE for RULE 2.

Let the baſe AB be 8, and the altitude CG be 3. Now $AC^2 = AG^2 + GC^2 = 16 + 9 = 25$; th. AC = 5. Th. $8 \times 6 + 5 \times 8 = 48 + 40 = 88$, the ſum to be multiplied; th. $88 \times 3 = 264$ the product; which, divided by 15, gives 17,6 the area of the ſegment ABC, nearly.

Problem 17.

367. *To find the area of an ellipſis, or oval.*

RULE *.

Multiply 0,7854, *the greateſt diameter, and the leaſt diameter together, and the product of theſe three numbers will expreſs the area.*

The reaſon of this rule (and of thoſe in articles 366 and 385, marked likewiſe with a ſtar) cannot be given from the preceding principles; but ſince they may prove very uſeful to artificers, it is hoped the more ſkilful reader will excuſe their being inſerted.

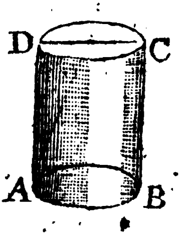
EXAMPLE.

What is the area of an ellipſis, whoſe greateſt diameter is 24, and leaſt diameter is 18?

Now the conſtant number is - - -	0,7854
- - - the greateſt diameter is - -	24 ×
	31416
	15708
The firſt product is - - -	18,8496
The leaſt diameter is - - -	18 ×
	1507968
	188496
The area is - - - - -	339,2928

Problem 18.

368. *To find the convex ſurface of a right cylinder* ABCD.



RULE.

Multiply the circumference of the baſe by the altitude of the cylinder, and the product will give the convex ſurface.

For conceiving the convex ſurface cut parallel to the axis, and then ſpread ſmooth upon a plain; it will evidently fall into a rectangular figure, and conſequently muſt be determined after the ſame manner. 345

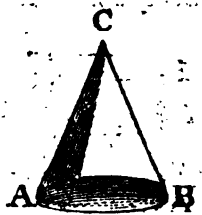
EXAMPLE.

What is the convex ſurface of a right cylinder, whoſe circumference is $9\frac{1}{2}$ feet, and length is $4\frac{1}{2}$ feet?

	1. Decimally.	2. By croſs multiplication
		f. : *i.*
Circumference - =	9,5	9 : 6
Length - - -	4,5	4 : 6
	475	4 : 9 : 0
	380	38 : 0
Convex ſurface =	42,75	= 42 : 9 : 0

Problem 19.

369. *To find the convex ſurface of a right cone* ABC.



RULE.

Multiply the circumference of the baſe by the ſlant ſide, and half the product will give the area.

For conceiving the ſurface cut in a ſtrait line from the vertex to the baſe, and then ſpread ſmooth upon a plain, it will evidently fall into the ſector of a circle; and conſequently, muſt be determined after the ſame manner - - - - - - - - 364

EXAMPLE.

What is the convex ſurface of a right cone, whoſe baſe is 64 feet in circumference, and ſlant ſide AC is 28 feet in length?

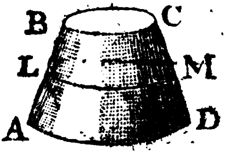
Now the circumference - - = 64
ſlant ſide AC - - - = 28 ×

512
128

The product - - - = 1792
The convex ſurface - - = 896 ſquare feet.

Problem 20.

378. *To find the convex ſurface of the fruſtum of a right cone* ABCD, *made by a ſection parallel to the baſe.*



RULE.

Multiply half the ſum of the circumferences of the ends by the ſlant ſide; and the product will give the convex ſurface.

For conceiving the convex ſurface cut in the ſtrait line AB, and then ſpread ſmooth upon a plain, it will evidently fall into the ſegment of a ſector; whoſe bounding arches are equal to the circumferences of the ends, and whoſe ſides are equal to the ſlant ſide of the fruſtum AB or DC; likewiſe the circumference of a circle [illegible] round the middle of the fruſtum, will fall into an arch LM biſecting the ſides of the ſegment; wherefore the convex ſurface is truly expreſſed by LM × AB. - - - - - - - - - 365

EXAMPLE.

Let the circumferences of the ends be 32, and 8 feet, and the length of the ſlant ſide AB be 7 feet; to find the convex ſurface. Now half $\overline{32+8} \times 7 = 20 \times 7 = 140$, the content required.

Problem 21.

371. *The diameter of a globe being given; to find the ſuperficies.*

RULE.

Find the circumference of a great circle upon the globe by article 356, multiply the circumference by the diameter, and the product will expreſs the ſuperficies.

Becauſe the ſurface of a globe is four times the area of its great circle (283) and the product of the circumference, by the diameter, is likewiſe four times the area of the ſame circle - - - - 362.

EXAMPLE.

What is the ſuperficies of a globe, whoſe diameter is 1?

Now the circumf. of a great circle = 3,1416. 355

the diameter - - - = 1 ×

The ſuperficies of the globe - = 3,1416

Problem 21.

372. *The diameter, or ſemidiameter of a globe being given, to find the ſuperficies.*

RULES.

1. *Multiply* 3,1416 *by the ſquare of the diameter, and the product will give the ſuperficies.* Or,

2. *Multiply the ſquare of the ſemidiameter by* 88, *divide the product by* 7, *and the quotient will give the ſuperficies.*

Becauſe the ſurfaces of globes, are proportional to the ſquares of their diameters, or ſemidiameters (290) and 3,1416, is the ſuperficies of a globe whoſe diameter is 1 - - - - - - - - 371

EXAMPLE.

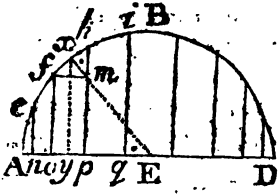
What is the ſuperficies of a globe whoſe diameter is 8 feet?

First method.		Second method.	
Conſtant No.	3,1416	Square of ſemidiam.	16
Square of diameter	64	Conſtant number	=88
	125664		128
	188496		128
Superficies	201,0624	Square feet.	7)1408(201

Problem

Problem 22.

373. *To find the ſuperficies of a ſegment of a globe, made by the ſection of a plain.*



RULE.

Multiply the circumference of the globe by the height of the ſegment, and the product will give the ſuperficies.

For let a ſemicircle ABD be divided into parts a e, e f, f h, &c. indefinitely ſmall, ſo as to be taken for ſtrait lines, biſect f h in x, draw e n, f o, x y, &c. perpendicular to AD, and f m parallel to AD, join the point x to the center E, and ſuppoſe the ſemicircle to turn about the axis AD and generate a globe.

Now E x f is a right angle (126) th. the angles x f m, E x y are equal, and the triangles h f m, E x y are equiangular; th. h f : f m :: E x : x y; but E x : x y :: the circumference of the globe c : the circumference of a circle, whoſe radius is y x (225) th. h f : f m :: c : circumference x (173) th. circumference x × h f = c × f m, or c × o p; but the ring, or zone deſcribed by f h, being the convex ſurface of the fruſtum of a right cone = circumference x × f h (370) th. the zone deſcribed by f h = c × o p (48). By the ſame reaſon, each zone deſcribed upon the globe, is equal to the product of the circumference c by the intercepted part of the axis, and conſequently a ſuperficies deſcribed by any arch A f, is equal to c × A o.

Corollary.

Corollary.

374. If the parts of the diameter a i, i g, g m, &c. are equal, the zones deſcribed by the correſponding arches a e, e f, f b, &c. are likewiſe equal.

Remark.

The ſuperficies of a globe is expreſſed by the product of its circumference and diameter, as before determined in article 371.

EXAMPLE.

What is the ſuperficies of a ſegment 9 feet high, cut from a globe of 42 feet diameter?

The conſtant number - - = 3,1416
The diameter of the globe - = 42

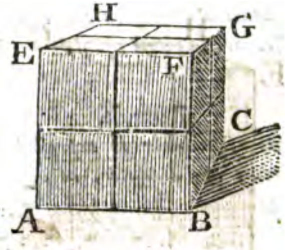
62832
125664

The circumference - - = 131,9472, by article 355.
The height of the ſegment - - - 9 ×
The ſuperficies required = 1187,5248

Problem.

Problem 23.

375. *To find the ſolidity or content of a cube* ABCH.



RULE.

Multiply the ſquare of a ſide by the ſide, and the product will expreſs the content.

For if a cube ABCH be compoſed or built up with leſſer cubes, the number of them placed upon the baſe is equal to the number of little ſquares in the baſe, and that number is expreſſed by AB × AB, the ſquare of the ſide (344). Again, the number of courſes is equal to the number of parts or meaſures in the ſide AE or AB; and conſequently the number of leſſer cubes contained in the greater, is expreſſed by AB×AB×AB, which gives the rule above.

EXAMPLE.

What is the content of a cube whoſe ſide is two feet?

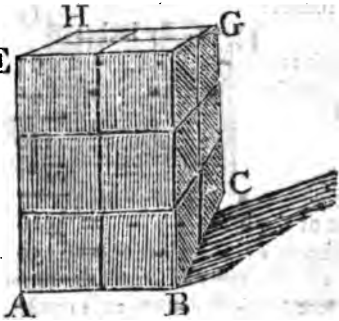
Now the ſide AB or AE - - = 2
- - - - - - - = 2

The ſquare of a ſide - - = 4 = AB×AB, or AB^2
Again, the ſide AB - - = 2

The content required - - = 8 = $AB^2 \times AB$.

Problem 24.

375. *To find the content of a priſm* ABCDEFGH.



RULE.

Multiply the area of the baſe by the height or altitude, and the product will expreſs the content.

For if a priſm is erected upon a ſquare baſe, and compoſed or built up with equal cubes, the number placed upon the baſe muſt be equal to the number of little ſquares in the baſe, and the number of courſes will be equal to the number of parts or meaſures in the altitude, wherefore the content is truly expreſſed as above in a ſquare priſm; and all priſms of equal baſe and altitude being equal (268) the rule is true univerſally.

EXAMPLE.

Let the area of the baſe ABCD be 4 feet, and the height AE be 3 feet, to find the content.

Now 4 × 3 = 12, the content required in cubic feet.

EXAMPLE 3.

What is the content of a block of marble, in length 7 feet 9 inches, breadth 3 feet 6 inches, and thickneſs 2 feet 6 inches?

1. By inches

The length is - - - - 93
The breadth is - - - - 42 ×

186
372

The area of the baſe - - = 3906
The thickneſs - - = 30 ×

The content - - = 117180 cubic inches, which divided by 1728 (the number of cubic inches in a cubic foot) gives 67,8 cubic feet, the content required.

2. By croſs multiplication.

f. i.
Again, the length is - 7 : 9
- - - - the breadth is - 3 : 6 ×

3 : 10 : 6
23 : 3

Area of the baſe - - 27 : 1 : 6
The thickneſs - - 2 : 6 ×

13 : 6 : 9 : 0
54 : 3 : 0

The content is - - 67 : 9 : 9 : 0 cubic feet.

Problem

EXAMPLE 3.

What is the content of a ſquare ciſtern, whoſe length, breadth, and depth are 54, 27, and 33 inches reſpectively?

Now the length - - = 54
- - - - - breadth - - = 27 ×

378
108

The product - - = 1458 the area of the bottom
The depth - - = 33 ×

4374
4374

The content - - = 48114 cubic inches, which, divided (by 282, the number of cubic inches in a gallon) gives 170 gallons, the content required.

EXAMPLE 4.

What is the content of a canal 20 feet broad at the top, 18 feet broad at bottom, 5 feet deep, and 300 feet long?

This ſolid being a priſm, whoſe ends or baſes are quadrangles having two parallel ſides, the area of the baſe (by article 348) = half $\overline{20+16}\times 5 = 18\times 5 = 90$, which, multiplied by 300, the length or height gives 90×300, or 27000 cubic feet, the content required.

Problem 25.

377. *To find the content of a wall, in ſquare rods of a brick and half thick.*

RULE.

Multiply the area of the wall in feet, by the number of half bricks in thickneſs, divide the product by 816, and the quotient will expreſs the content required.

Becauſe $15\frac{1}{2} \times 15\frac{1}{2} \times 3$, or $272\frac{1}{4} \times 3$, or $816\frac{3}{4}$, expreſſes the content of a wall that is a ſquare rod in area, and three half bricks in thickneſs (376); which ſolid is called a ſquare rod of work; the fraction $\frac{3}{4}$ being rejected in favour of the workmen.

EXAMPLE.

How many ſquare rods of work is there in a wall $187\frac{1}{2}$ feet long, 7 feet high, and $2\frac{1}{2}$ bricks thick?

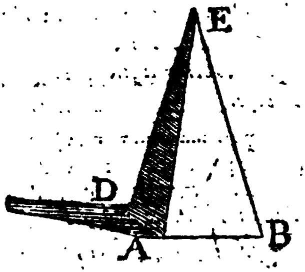
Now the length of the wall is 187,5
- - - - - - height - - - 7 ×

The area of the wall is - - 1312,5
Multiply by - - - - - 5 in thickneſs

The number to be divided is 6562,5, which, divided by 816 gives 8 rods, the content required.

Problem 26.

378. *To find the content of a pyramid* ABCDE.



RULE.

Multiply the area of the baſe, by a third part of the altitude, and the product will give the content.

Becauſe a pyramid is a third part of a priſm, having the ſame baſe and altitude - - - - 275

EXAMPLE.

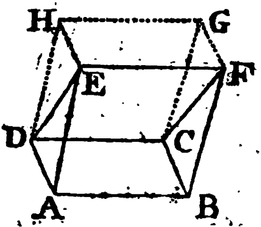
What is the content of a pyramid 300 feet high, erected upon a baſe 24 feet ſquare?

Now the ſide of the baſe is - - 24
× 24
96
48
The area of the baſe - - - - = 576
One third of the height - - = 100
The content - - - = 57600 cubic feet.

Problem

Problem 27.

379. *To find the content of a wedge* ABCDEF, *which is a ſolid contained under five plains; the back or baſe* ABCD, *is a rectangle or oblong, and the four ſides terminate in the edge* EF, *being a ſtrait line parallel and equal to a ſide of the baſe.*



RULE.

Multiply the area of the baſe, by half the altitude of the edge, and the product will give the content.

For this ſolid is likewiſe a triangular priſm, whoſe baſes are ADE, BCF; and if plains be drawn through AE, DC parallel to AC, AF, the ſolid ABCDEFGH formed thereby will alſo be a priſm; and either the rectangle AC, or parallelogram AH, may be taken for its baſe (234). Th. the priſm ADEBCF = HEDGFC (268). Th. ADEBCF, or ABCDEF is half ABCDEFGH, which gives the rule above.

EXAMPLE.

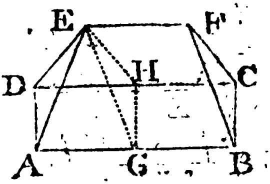
What is the content of a wedge whoſe baſe meaſures 36 feet by 20, and whoſe height is 12 feet?

Now 36 × 20 = 720, the area of the baſe. Th. 720 × 6 = 4320, the content required.

Problem

Problem 28.

380. *To find the content of a pavilion roof* ABCDEF, *which is a ſolid contained under five plains; the baſe is a rectangle or oblong, and the four ſides terminate in a ridge* (EF) *parallel to a ſide of the baſe, but unequal to it.*



RULE.

To the length of the ridge, add twice the ſide of the baſe which is parallel to it. Multiply the ſum by the other ſide of the baſe, and the product which ariſes by a ſixth part of the altitude, and the ſecond product will give the content.

For ſuppoſing the ſection EGH, made parallel to the plain FBC, the roof is then divided into the pyramid AGHDE, and the wedge GBCHEF; now calling the altitude a, and finding the contents of thoſe parts according to the preceding rules, their ſum (properly ordered) will be expreſſed by $\overline{EF+2AB}\times BC\times\frac{a}{6}$, which gives the rule above.

EXAMPLE.

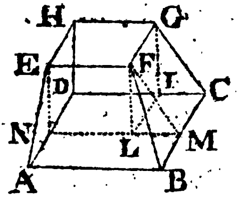
What is the content of a pavilion roof, whoſe baſe is 16 feet by 20, ridge parallel to the greater ſide is 16, and altitude is 12 feet?

Now 16 + 72 = 88, the ſum to be multiplied; th. 88 × 20 × 2 = 3520 cubic feet, the content required.

Problem

Problem 29.

281. *To find the content of the fruſtum of a ſquare pyramid* ABCDEFGH, *made by a ſection parallel to the baſe.*



RULE.

To the areas of the ends, add the product of their ſides, multiply the ſum by a third part of the altitude, and the product will give the content.

For let the ſections FGIL, EFMN be made parallel to the plains AH, HC, and the fruſtum will be divided into the priſm NLIDEFGH, the wedge MCILFG, and the pavilion roof ABMNEF; now calling the altitude a, and finding the contents of thoſe parts according to the preceding rules, their ſum (properly ordered) will be expreſſed by $\overline{AB^2+EF^2+AB\times EF}\times\frac{a}{6}$, which gives the rule above.

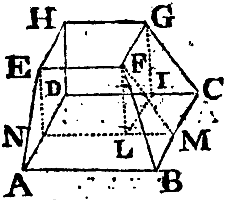
EXAMPLE.

What is the content of the fruſtum of a pyramid 60 feet high, whoſe ends are 16 and 13 feet ſquare?

Now 16×16 = 256, 13 × 13 = 169, and 16 × 13 = 208, which are the areas of the ends, and the product of their ſides; th. 256 + 169 + 208 = 633, the ſum to be multiplied; th. 633 × 20 = 12660, the content required.

Problem 50.

382. *To find the content of a priſmoid* ABCDEFGH, *being a ſolid contained under ſix plains; the baſes or ends are parallel rectangles or oblongs, and the four ſides are quadrangles.*



RULE.

To the areas of the ends, add the product of the ſums of their lengths and breadths; multiply this ſum by a ſixth part of the altitude, and the product will give the content.

For let the ſections FGIL, EFMN be made parallel to the plains AH, HC, and the priſmoid will be divided into the priſm NLIDEFGH, the wedge MCILFG, and the pavilion roof ABMNEF; now calling the altitude a, and finding the contents of theſe parts by the preceding rules, their ſum (properly ordered) will be expreſſed by $\underline{AB \times BC} + \underline{EF \times FG} + \underline{AB + EF} \times \underline{BC + FG} \times \frac{a}{6}$, which gives the rule above.

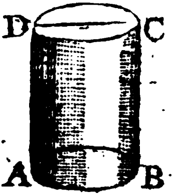
EXAMPLE.

What is the content of a canal 304 feet by 20 at top, 300 feet by 16 at bottom, and 5 feet deep?

Now 304 × 20 = 6080, 300 × 16 = 4800, and 604×36=21744, which are the numbers to be added; th. 6080 + 4800 + 21744=32644, the ſum to be multiplied; th. 32624 × $\frac{5}{6}$ = 27186 the content required.

Problem 31.

383. *To find the content of a cylinder* ABCD.



RULE.

Multiply the area of the baſe by the altitude of the cylinder, and the product will expreſs the content.

For all priſms and cylinders of equal baſe and altitude are equal (268); and therefore muſt be determined by the products of their baſes and altitudes. 376

EXAMPLE.

What is the content of a cylinder 3 feet diameter, and 6 feet high?

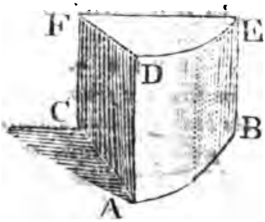
Now 3 × 3 = 9, the ſquare of the diameter.

Th. 14 : 11 :: 9 : 7,07, the area of the baſe, by rule 3, article 362.

Th. 7,07 × 6 = 42,42 the content required.

Problem 91.

384. *To find the content of a triangular ciſtern, whoſe bottom is the ſector of a circle.*



RULE.

Multiply the area of the bottom in inches, by the depth in inches; divide the product by 282, and the quotient will be the content in gallons.

For this ſolid is evidently a portion of a cylinder, and conſequently muſt be determined by a ſimilar rule.

EXAMPLE.

What is the content of a ceſtern whoſe bottom is a quarter of a circle 21 inches in ſemidiameter, and whoſe depth is 42 inches?

Now by rule 4, article 362, the area of a circle 21 inches in ſemidiameter is 1386, its fourth part 346,5 is the area of the bottom of the ceſtern, and 346,5 × 42 = 14553 the content in cubic inches, which, divided by 282 gives 51,6 gallons, the content required.

Problem 338.

385. *The bung diameter, head diameter, and length of a caſk (within ſide) being given; to find the content of a cylinder nearly equal to it; which is called gauging the caſk.*

RULE.

To the head diameter, add ſeven tenth parts of the difference between the bung and head diameters, and the ſum will be a mean diameter of the caſk, or the diameter of a cylinder equal to the caſk. Multiply the ſquare of the mean diameter, the length of the caſk, and 0,78 together; and the product will be the content, nearly.

Note. The number ($\frac{7}{10}$) uſed in finding a mean diameter, is thought the beſt adapted to a general rule.

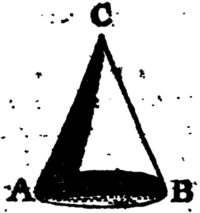
EXAMPLE.

What is the content of a caſk, whoſe bung diameter, head diameter, and length, are 32, 26 and 40 inches, (within ſide) reſpectively?

Now 32 − 26 = 6, the difference of diameters, and 6 × 0,7 = 4,2 the number to be added; th. 26 + 4,2 = 30,2 the mean diameter, whoſe ſquare is 30,2 × 30,2 or 912,04; th. 912,04 × 40 × 0,78 = 28455 the content in cubic inches, which, divided ſeperately by 282 and 231, will give 101 ale, and 123 wine gallons, the contents required.

Problem 34.

386. *To find the content of a cone* ABC.



RULE.

Multiply the area of the baſe by a third part of the altitude, and the product will give the content.

Becauſe a cone is a third part of a cylinder, having the ſame baſe and altitude.

EXAMPLE.

What is the content of a cone whoſe baſe is 3 feet diameter, and altitude is 6 feet?

Now 3 × 3 = 9 the ſquare of the diameter of the baſe.

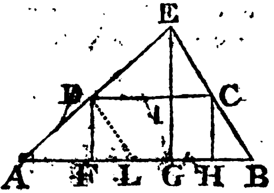
And 14 : 11 :: 9 : 7,07, the area of the baſe, by rule 3, article 462.

Then 7,07 × 2 = 14,14, the content required.

Problem

Problem 35.

387. *To find the content of the frustum of a cone* ABCD, *made by a ſection parallel to the baſe.*



RULES.

To the ſquares of the {*diameters* / *circumferences*} *of the ends add their product; multiply the ſum by the altitude of the fruſtum, and the product which ariſes by* {0,2618 / 0,0265} *and this laſt product will give the content.*

For ſuppoſe E the vertex of the compleat cone, and the triangle ABE a ſection thro' the axis; draw DL parallel to EB, and let fall the perpendiculars DF, EG, CH. Now LB is equal to DC (91) and the triangles ALD, ABE, DCE are equiangular (83); th. AL : DF :: AB : EG :: DC : EI (201); whence the altitudes EG, EI are determined; now calling the diameters of the ends D, d, their circumferences C, c; and finding the contents of the cones ABE, DCE, by article 386; their difference will produce the following expreſſions, namely,

1. $ABCD = \overline{D^2 + d^2 + D \times d} \times DF \times 0{,}2618.$

2. $ABCD = C^2 + c^2 + C \times c \times DF \times 0{,}0265$, which give the rules above.

EXAMPLE.

EXAMPLE 1.

What is the content of the fruſtum of a cone 60 feet high, the diameters of its ends being 20 and 3 feet?

Now 20 × 20 = 400, 3 × 3 = 9, and 20 × 3 = 60, which are the ſquares of the diameters and their product. Th. 400 + 9 + 60 = 469, the ſum to be multiplied.

The altitude - - - = 60

First product - - 28140
× 0,2618

225120
28140
168840
56280

Content - - - = 7366,0520

EXAMPLE 2.

What is the content of a conical fruſtum, the circumferences of whoſe baſes are 66 and 56 feet, and whoſe height is 4 feet?

Here 66 × 66 = 4356, 56 × 56 = 3136, and 66 × 56 = 3696; which are the ſquares of the circumferences, and their product. Th. 4356 + 3136 + 3696 = 11188 the ſum to be multiplied by - - - 4

The firſt product is - - - 44752
× 0,0265

223760
268512
89504

The content required - - - = 1185,9280

Problem

Problem 36.

To meaſure TIMBER.

388. A ſquare piece of timber equally thick at both ends is a priſm, a round piece equally thick at both ends is a cylinder; a ſquare piece that tapers regularly is the fruſtum of a pyramid, and a round piece that tapers regularly is the fruſtum of a cone; and the contents of theſe ſolids may be exactly computed by their reſpective rules.

But becauſe the menſuration of tapering timber by the exact rules is troubleſome, an approximation has taken place, and the contents of ſuch trees are generally computed by the following

RULE.

Multiply the ſquare of the girt in inches by the length in feet, divide the product by 144, and the quotient will give the content in feet.

Remarks.

1. *The girt of a piece of timber is a fourth part of its compaſs or circumference at the middle.*

2. *Trees of irregular growth muſt be meaſured in parts or pieces, as above directed.*

3. *Allowance muſt be made for the thickneſs of bark, if on the tree.*

EXAMPLE 1.

What is the content of a tree whoſe girt is 16 inches, and length is 30 feet?

Now 16 × 16 - - - = 256 the ſquare of the girt
Multiply by - - - 30 the length

The product is - - 7680, which, divided by 144 gives 53,3 cubic feet, the content required.

EXAMPLE 2.

What is the content of a tree whoſe girt is 13 inches, and length is 40 feet 6 inches?

Now 13 × 13 - - = 169 the ſquare of the girt
The length - - = 40,5

845
- - - - - - 676

The product - - = 6844,5, which, divided by 144, gives 47,5 the content required.

EXAMPLE 3.

What is the content of a piece of timber whoſe girt is 14 inches, and length is 20 feet?

Now 14 × 14 = 196 the ſquare of the girt, and 196 × 20 = 3920 the product; which, divided by 144 gives 27,2 the content required.

Problem 37.

389. *The diameter of a globe being given to find the ſolidity, or content.*

RULES.

1. *Find the ſuperficies by the* 373, *multiply the ſuperficies by a third part of the ſemidiameter, and the product will give the content.*

Becauſe a globe is equal to a pyramid or cone, whoſe baſe is equal to the ſurface, and altitude is equal to the ſemidiameter - - - - - 270

2. *Find the content of a circumſcribing cylinder by the* 383, *and take two thirds of it for the content of the globe.*

For a globe is two thirds of its circumſcribing cylinder - - - - - - - - 281

EXAMPLE.

What is the content of a globe whoſe diameter is 1?

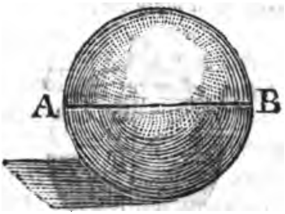
1. The ſuperficies - - - - - = 3,1416
One third of the ſemidiameter - - = $\frac{1}{6}$
The content - - - - - - = 0,5236

2. The content of a circumſcribing cylinder is 0,7854, and $\frac{2}{3}$ of it is 0,5236, the content of the globe, as before.

Again,

Again, Problem 37.

390. *The diameter of a globe being given, to find the content.*



RULE.

Multiply the cube of the diameter by 0,5236, *and the product will give the content.*

For globes are proportional to the cubes of their diameters (by 291), and 0,5236 is the content of a globe whoſe diameter is 1 - - - - - 389

EXAMPLE.

What is the content of a globe whoſe diameter AB is 12 inches?

Now 12 × 12 × 12 - - = 1728 the cube of the diameter - - - 0,5236 ×

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 10368
 5184
3456
8640
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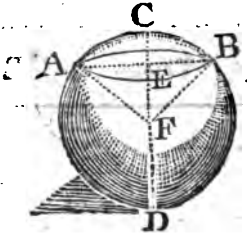
The content - - = 904,7808 cubic inches.

Problem

Problem 38.

391. *To find the content of a ſegment of a globe* ABC, *made by the ſection of a plain* AB.



RULE.

From three times the diameter of the globe, take twice the altitude of the ſegment; multiply together the remainder, the ſquare of the altitude, and 0,5236; and this product will give the content.

For the conical ſolid ACBF being equal to a pyramid of equal baſe and altitude (270) may be expreſſed in the given terms, and the cone ABF having the ſame baſe as the ſegment may likewiſe be expreſſed in the given terms, by what preceeds; and the difference of theſe expreſſions gives $\overline{3CD - 2CE} \times CE^2 \times 0{,}5236$, the rule above.

Note. If the diameter of the ſegment AB is given, make CE : EA :: AE : ED, and CD will be known.

EXAMPLE.

What is the content of a ſegment 5 feet high, cut from a globe 20 feet diameter?

Now $60 - 10 = 50$ the remainder, and $5 \times 5 = 25 = CE^2$.

Th. $50 \times 25 \times 0{,}5236 = 654{,}5$ the content required.

A TABLE *ſhewing the weight of a cubic foot, of various bodies, in pounds averdupoiſe.*

Things.	lb.	oz.
Gold	1227	8
Quickſilver	860	2
Lead	707	1
Silver	693	3
Copper	548	1
Steel	490	10
Bar iron	485	4
Block tin	455	6
Caſt iron	445	5
Marble	169	6
Portland ſtone	160	10
Freeſtone	147	0
Loom	135	0
Brick	125	0
Gravel	109	5
Clay	107	0
Chalk	67	7
Salt water	64	6
Rain water	62	8
Salt	43	12

Mahogar